00063762

NOT TO BE ISSUED OUT
WITHOUT PERMISSION.

O
620
Wes/Sim
63762

THE
ASIATIC SOCIETY OF BOMBAY
Town Hall, Bombay.

Digitized with financial assistance from
Observer Research Foundation
on 22 February, 2019

SIMPLE EXPLANATIONS

OF

ENGINEERING FORMULÆ.

Direct Stress and Stiffness.

SIMPLE EXPLANATIONS

OF

ENGINEERING FORMULÆ.

Direct Stress and Stiffness.

B 949

63762

BY

R. W. WESTERN.

LIBRARY OF THE BOMBAY BRANCH OF THE ROYAL ASIATIC SOCIETY

LONDON:
B. T. BATSFORD, 52, HIGH HOLBORN.
1892.

C
62.C
Wec/sim
63762

LONDON:
PRINTED BY WERTHEIMER, LEA AND CO.
CIRCUS PLACE, LONDON WALL, E.C.

PREFACE.

MANY writers on this subject ostensibly avoid any use of the calculus, as likely to scare their readers, however, they cannot help using the method of the calculus, since most of these formulæ are quite inexplicable without it; hence, those tortuous and long winded discourses, generally about Δ instead of d which go through exactly the same line of reasoning and take twice the time about it. Their readers are informed that no knowledge of higher mathematics will be required, and fancy such a work will be the easier on that account—an utter delusion.

The symbols $\int$ and dx may be dispensed with, but the ideas they represent cannot.

Suppose a governess, knowing that any direct allusion to the four rules of arithmetic, was distasteful to her pupils, were to teach them to find the value of 9 articles at $3s.\ 6d.$ a piece, by counting the coins, or on their fingers. When it was desired to know the value of 3 articles at $10s.\ 6d.$, the same process would be necessary.

A nice equipment for the arithmetical exigences of life.

As a matter of fact, a straightforward use of the calculus is by far the easiest way of mastering that modicum of theory, which no engineer can any longer afford to dispense with.

It is astonishing how little of the calculus is necessary, how easy that is to acquire, and what an immense simplification it brings about.

For the purposes of this book but one formula of the integral calculus is required, and that is sufficiently explained in the introduction.

The reader is assumed to be acquainted with elementary algebra, and mechanics; so as to be able to find the general expression for bending moment, or shearing force at any section of a beam, in the manner described in all text books on the subject.

SIGNIFICATION OF SYMBOLS.

a = an amount of alteration in length, due to pressure.
b = the breadth of a figure.
D = a distance ; or the depth of a girder.
E = the modulus of elasticity.
F = a shearing force.
f = a proof stress.
h = the height of a figure.
I = moment of inertia.
i = a strain.
k = radius of gyration.
L = the length of anything.
l = the span of a beam.
M = a bending moment.
N = a number.
n = a number.
P = a force, or pressure.
p = a stress, or intensity of pressure.
p_o = a particular value of p.
p_1 = a working stress.
R = the radius of curvature ; or the resilience of a body.
S = the area of a cross section.
t = the thickness of anything.
V = the volume of material in a body.
v = the distance between two points in a section.
x = the distance of any point in a figure from a given axis or origin.
X = the greatest value of x.
y = the distance of any point in a figure from a given axis or origin, when measured in direction at right angles to that of x.
y_o = a particular value of y.
y_1 = the value of a y which corresponds with the attainment of working stress in the structure.
δ = the value of y which corresponds with proof stress.

CONTENTS.

INTRODUCTION - - - - - - - - PAGE 1

CHAPTER I.

THE RELATION OF STRESS TO STRAIN.

$$E = \frac{\frac{P}{S}}{\frac{a}{L}} = \frac{p}{i} = \frac{P}{a}\frac{L}{S}.$$

PAGE 6

CHAPTER II.

THE RESILIENCE OF STRAIGHT BARS.

$$R = \frac{V}{E} \cdot \frac{f^2}{2} = l\frac{Q}{2}.$$

PAGE 9

CHAPTER III.

THE RELATION BETWEEN BENDING MOMENT AND SHEARING FORCE.

$$M = \int_o^x F\,dx.$$

$$\frac{dM}{dx} = F.$$

PAGE 11

CHAPTER IV.

THE DISTRIBUTION OF STRESS ON THE CROSS SECTION OF A BEAM.

$$p = E\frac{x}{R}.$$

$$p_o = E\frac{X}{R}.$$

PAGE 13

CHAPTER V.

THE MOMENT OF RESISTANCE OF A BEAM.

$$M = \frac{p}{x} I = \frac{p_o}{X} I.$$

PAGE 16

CHAPTER VI.

To Find the Moment of Inertia of an Oblong Section.

$$I = \frac{h^3 b}{12}.$$ PAGE 19

CHAPTER VII.

To Find the Moment of Inertia of a Symmetrical I Section.

$$I = \tfrac{1}{12}\left\{ h^3 b - (h - 2tf)(b - tw) \right\}.$$ PAGE 20

CHAPTER VIII.

Moment of Inertia about any Axis.

$$I_m = I_n + D^2 S.$$ PAGE 21

CHAPTER IX.

Radius of Gyration.

$$k = \sqrt{\frac{I}{S}}.$$ PAGE 25

CHAPTER X.

The Distribution of Stress upon a Section of any Structure, when it is acted upon by Bending Moment, in Combination with Forces which produce Uniform Normal Stress.

$$p = p_u \pm \frac{M}{I} x.$$

$$p_o = p_u \pm \frac{M}{I} X.$$ PAGE 26

CHAPTER XI.

The Distribution of Stress upon a Section of any Structure, when it is acted upon by a Normal Force, the Line of Action of which does not pass through the Centre of Gravity of the Section.

$$p = \frac{P}{S}\left(1 \pm \frac{vx}{k^2}\right).$$

$$p_o = \frac{P}{S}\left(1 \pm \frac{v X}{k^2}\right).$$ PAGE 30

CHAPTER XII.

THE STABILITY OF WALLS AND DAMS.

$$v = \tfrac{1}{6}\,\mathrm{T}.$$

PAGE 33

CHAPTER XIII.

TO CUT THE STRONGEST RECTANGULAR BEAM OUT OF A STRAIGHT ROUND LOG.

$$b = \cdot 57\,\mathrm{D}.$$

$$h = b\sqrt{2}.$$

PAGE 35

CHAPTER XIV.

THE NUMBER OF TURNS WHICH MAY BE MADE WITH SAFETY IN WINDING UP A SPIRAL SPRING.

$$\mathrm{N} = \frac{p_1\,\mathrm{L}}{\pi\,\mathrm{E}\,t}.$$

PAGE 38

CHAPTER XV.

TO FIND THE MOMENT EXERTED BY A SPIRAL SPRING AFTER N TURNS OF THE AXLE HAVE BEEN MADE.

$$\mathrm{M} = 2\,\pi\,\mathrm{N}\,\frac{\mathrm{E\,I}}{\mathrm{L}}.$$

PAGE 41

CHAPTER XVI.

THE RESILIENCE OF A SPIRAL SPRING OF RECTANGULAR SECTION.

$$\mathrm{R} = \tfrac{1}{6} f^2\,\frac{\mathrm{V}}{\mathrm{E}}.$$

When the spring is of circular section.

$$\mathrm{R} = \tfrac{1}{8} f^2\,\frac{\mathrm{V}}{\mathrm{E}}.$$

PAGE 42

CHAPTER XVII.

THE DIFFERENTIAL EQUATION TO THE DEFORMED MEAN FIBRE OF A BEAM.

$$d\,\frac{dy}{dx} = \frac{\mathrm{M}}{\mathrm{E\,I}}\,dx.$$

PAGE 44

CHAPTER XVIII.

THE DEFLECTION OF A BEAM DUE TO A SINGLE WEIGHT PLACED AT THE CENTRE.

$$y_o = \tfrac{1}{48} \frac{W\, l^3}{E\, I}.$$

$$y_1 = \tfrac{1}{192} \frac{p_1\, l^2}{X\, E}.$$

$$R = \frac{y_o}{2} W = \tfrac{1}{6} \frac{f^2\, l}{E X^2} I. \qquad \text{PAGE } 47$$

CHAPTER XIX.

THE DEFLECTION OF A CANTILEVER DUE TO A SINGLE WEIGHT PLACED AT THE FREE END.

$$y_o = \tfrac{1}{3} \cdot \frac{W\, l^3}{E\, I}.$$

$$y_1 = \tfrac{1}{3} \cdot \frac{p_1\, l^2}{X\, E}.$$

$$R = \frac{y_o}{2} W = \tfrac{1}{6} \cdot \frac{f^2\, l}{E X^2} I. \qquad \text{PAGE } 51$$

CHAPTER XX.

THE DEFLECTION OF A CANTILEVER DUE TO A LOAD UNIFORMLY DISTRIBUTED.

$$y_o = \tfrac{1}{8} \frac{W\, l^3}{E\, I}.$$

$$y_1 = \tfrac{1}{4} \frac{p_1\, l^2}{X\, E}. \qquad \text{PAGE } 54$$

CHAPTER XXI.

THE DEFLECTION OF A BEAM DUE TO A LOAD UNIFORMLY DISTRIBUTED.

$$y_o = \tfrac{5}{384} \cdot \frac{W\, l^3}{E\, I}.$$

$$y_1 = \tfrac{5}{48} \cdot \frac{p_1\, l^2}{X\, E}. \qquad \text{PAGE } 57$$

CHAPTER XXII.

THE SAFE WORKING DEFLECTION FOR BUILT GIRDERS OF UNIFORM DEPTH.

$$y_1 = \frac{p_1 l^2}{4 E D}.$$

$$\frac{D}{l} = 400 \frac{p_1}{E}.$$ PAGE 62

CHAPTER XXIII.

THE DEFLECTION OF CARRIAGE SPRINGS.

$$y_o = \frac{3 W l^3}{n E t^3 b}.$$ PAGE 65

CHAPTER XXIV.

THE THEOREM OF THREE MOMENTS.

$$4 M_a l_1 + 8 M_b (l_1 + l_2) + 4 M_c l_2 = - w_1 l_1 - w_2 l_2.$$ PAGE 67

CHAPTER XXV.

EXPANSION ROLLERS.

$$W^2 = \tfrac{e^4}{9} p_o^3 \frac{l^2 R^2}{E}.$$

$$R = \frac{3}{8 l} \cdot \sqrt{\frac{W E}{p_o^3}}.$$ PAGE 73

All these formulæ are in their most general form, and may therefore be used with any system of weights and measures, English or foreign. It is only necessary to see that the numerical equations derived from them, are homogeneous.

For example :—

The formula, $M = \frac{p_o}{X} I$; proved in Chapter V. becomes $M = p_o \frac{h^2 b}{12}$ when applied to a beam of rectangular section.

Now M the bending moment on the cross section is generally obtained in *foot lbs.*; whereas h and b are generally given in inches and p_o in tons per square inch.

So that we have

$$\text{foot lbs} = \frac{\text{tons}}{(\text{inches})^2} \times \frac{(\text{inches})^2\ (\text{inches})}{\text{a number}}.$$

Whence $$\text{foot lbs.} = \text{inch tons}$$

which is nonsense. M the bending moment in foot tons must be divided by 12 × 2,240 to make it inch tons, and then the equation holds good.

It is well in many calculations to show the homogeneity as above.

Thus, the example in Chapter XX. is an application of the formula

$$y_o \frac{1}{8} \frac{W\ l^3}{E\ I}.$$

W	is given in	...	...	...	cwt.
l	,,	...	...	...	feet.
E	,,	...	...	...	tons per square inch.
I	,,	...	...	...	inches.

Here the weight 8 cwt. is divided by 20 to make it tons; l the length is multiplied by 12 to make it inches. I being made up of (inches) × (inches) × (inches) × (inches), is $(\text{inches})^4$.

So that we have

$$y_o = (\text{a number}) \times \frac{\text{tons} \times (\text{inches})^3}{\frac{\text{tons}}{(\text{inches})^2} \times (\text{inches})^4}.$$

Whence $$y_o = \text{inches}$$

the answer therefore will be in inches.

SIMPLE EXPLANATIONS
OF
ENGINEERING FORMULÆ.

Direct Stress and Stiffness.

INTRODUCTION.

HOWEVER small two quantities may be, there is no reason why one of them should not be two or three times larger than the other. For instance, a specimen of brass may be composed of two parts of copper to one of zinc; then though a particle, as small as possible, of this specimen be taken we know the amount of copper in it will still be twice as much as the zinc. In other words the ratio of the copper to the zinc in any particle of the specimen, *however small*, is 2. It will be granted then, that although two quantities be both of them infinitesimally small, they may at the same time have a definite and finite ratio.

Now let us consider the equation $m = x^2$. In such an equation m and x may denote whatever you please, provided that m is equal to x^2. For instance, you cannot make m stand for shillings and x for pounds, because $shillings = (pounds)^2$ is nonsense, which ever way you look at it; but x might measure the side of a square and m its area, or x might measure the circumference of a rope and m its breaking load. Suppose the thing, or whatever is represented by x is increased a very little bit, *as small as possible*, so that it can no longer be denoted by x, but must be denoted by $(x + dx)$, when dx stands for the little bit added on. Very well then, as the thing represented by x has been increased a little, the thing represented by m will also have been increased, and we must now denote it by $(m + dm)$, where again dm stands for the little bit by which it has been increased. Thus the equation becomes

$$m + dm = (x + dx)^2,$$

then by algebra

$$m + dm = x^2 + 2x(dx) + (dx)^2.$$

But since $m = x^2$, these two will cancel, so that we get

$$dm = 2x(dx) + (dx)^2.$$

Do not mix up the expression dx as used here, with $d \times x$, which it would signify in simple algebra. In this department of mathematics,

the symbol dx means such a small additament to x as has been just described.

It will be remembered that we said dx was to be as small as possible, from this it would appear that $(dx)^2$ is smaller than possible. This is intelligible when we consider, that if dx was so large a quantity as ·00001 of x; $(dx)^2$ would be ·0000000001; however small dx is made, $(dx)^2$ must be as small again. Thus when dx is infinitely small, $(dx)^2$ may be left out altogether without affecting the accuracy of the result. If any one further doubt the justice of this proceeding, let him consider that the whole structure of modern mathematics is founded upon it.

The above equation may therefore be written $dm = 2\,x\,dx$,

whence it follows $$\frac{dm}{dx} = 2\,x.$$

Here are two infinitesimal quantities with a ratio of $2\,x$; but it must be borne in mind that they cannot have this ratio unless they *are* infinitesimal. Since $m = x^2$, we may write $d\ (x^2)$ for dm, and then,

$$\frac{d\ (x^2)}{dx} = 2\,x.$$

We have in fact differentiated (as it is called) x^2, and $2\,x$ is the differential coefficient of x^2.

Similarly the differential coefficient of cx^2 is $2\,cx$, c being a constant or number. For taking as our original equation $m = cx^2$ and proceeding as before $$(m + dm) = c\,(x + dx)^2$$

and $$m + dm = c\,x^2 + 2\,cx\,(dx) + c\,(dx)^2,$$

m cancels $c\,x^2$ in the other side of the equation, and we may reconcile ourselves to cutting out $c\,(dx)^2$ by reflecting that if $(dx)^2$ is nothing, multiplying it by c can make it no larger.

So we get $$dm = 2\,c\,x\,dx \quad \ldots\ldots\ldots \quad (I)$$

from which $$\frac{dm}{dx} = 2\,c\,x,$$

or $$\frac{d\ (c\,x^2)}{dx} = 2\,c\,x.$$

Thus $2\,c\,x$ is the differential coefficient of $c\,x^2$; this includes the former case, because when $c = 1$ we have again $2\,x$ the differential coefficient of x^2.

But let us consider the equation marked (I) $dm = 2\,c\,x\,dx$. If ever we came across such an equation as this again, we should know at once by reasoning backwards that it follows that $m = c\,x^2$. Now here is an instance of how we might meet such an equation. We know that if the radius of a circle is r, its circumference will be $2\,\pi\,r$, when $\pi = 3{\cdot}14159$, etc. This fact was not discovered without some trouble, and indeed constitutes the famous problem of "squaring the circle." But

knowing that the circumference of the circle is $2\pi r$, how shall we set about to find the area of the circle? Take m to denote the area of the circle; from the same centre describe another circle (Fig. 1) close to the former, as close as possible; then these two circles being very close, we may call their distance apart dr, and the area between them dm. Now dm is a very thin strip, $2\pi r$ long, and dr broad. Therefore, $dm = 2\pi r\,(dr)$. But this is our old friend marked (I), π standing instead of c and r for x. So we know at once that $m = \pi r^2$ and that this is the area of the circle.

The operation just performed is called "integrating," and is written thus $\int 2\pi r\,dr$ or $2\pi \int r\,dr = \pi r^2$,

quantities like 2 and π being generally placed outside the $\int$, or sign of

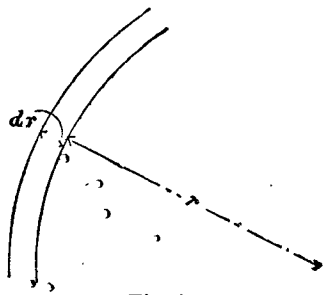

Fig. 1.

integration, since they are not affected by it. Thus we derive the formula:

$$c\int x\,dx = \frac{c}{2}x^2.$$

Now let us consider what is involved in the transition from equation $dm = 2\pi r\,dr$ to equation $m = \pi r^2$; dm means simply a little bit of the area m, and m might be regarded as made up of an infinite number of dm's, all of which would have to be added up to arrive at the value of m. Then considering the other side of the equation $2\pi r\,dr$, we have seen that it represents a thin ring. If we imagine r to start very small and grow, to start in fact from nothing and to gradually grow to its full size in the circle, the thin ring $2\pi r\,dr$ will grow at the same time, and if all the different sizes through which the ring passed while growing are added up, we shall get the whole area of the circle. This is why we sometimes write $\pi r^2 = 2\pi \int_0^r r\,dr$,

indicating that we must consider whatever r represents to have passed

from nothing up to r. This is important, for suppose you did not want to add the rings up all the way to the outer circumference, but only half way for instance ; we should have

$$2\pi\int_0^{\frac{r}{2}} r\,dr = \pi\left(\frac{r}{2}\right)^2,$$

the area of a circle with half the radius.

Consider now the equation $m = c\,x^3$. Proceeding as before

$$m + dm = c\,(x + dx)^3,$$

then by algebra

$$(m + dm) = c\,x^3 + 3\,c\,x^2\,(dx) + 3\,c\,x\,(dx)^2 + c\,(dx)^3$$

m cancels with $c\,x^3$, and if $3\,c\,x\,(dx)^2$ is negligible $c\,(dx)^3$ must be more so.

Hence

$$dm = 3\,c\,x^2\,dx$$

$$\frac{dm}{dx} = 3\,c\,x^2.$$

Thus the differential coefficient of $c\,x^3$ is $3\,c\,x^2$

Also returning from the equation

$$dm = 3\,c\,x^2\,dx,$$

to the equation $m = c\,x^3$,

we see the integral of $3\,c\,x^2\,dx$ is $c\,x^3$

or

$$3\,c\int x^2 dx = c\,x^3,$$

from which we derive the formula

$$c\int x^2\,dx = \frac{c}{3}\,x^3,$$

true as before for all values of c including unity.

It is easy to see that the differential coefficient of $c\,x$ is c. For if

$$m = cx$$

and

$$m + dm = c\,(x + dx),$$

$$dm = c\,dx$$

so that

$$\frac{dm}{dx} = c,$$

from which it is clear that $c\int dx = c\,x$.

Let us now collect and compare the results we have obtained so far :—

The differential coefficient of $c\,x$ is c ; and $c\int dx = c\,x$.

" " " $c\,x^2$ is $2\,c\,x$; and $c\int x\,dx = \frac{c}{2}\,x^2$.

" " " $c\,x^3$ is $3\,c\,x^2$; and $c\int x^2\,dx = \frac{c}{3}\,x^3$.

It might be proved, in the same manner, that the differential coefficient of

$$c\,x^4 \text{ is } 4\,c\,x^3\text{; and, conversely, } c\int x^3\,dx = \frac{c}{4}x^4.$$

Then considering the regularity of these results you would be right in guessing that the differential coefficient of

$$c\,x^n \text{ is } c\,n\,x^{n-1}\text{; and } c\int x^n\,dx = \frac{c}{n+1}\,x^{n+1},$$

when n is any number.

These last two formulæ include of course all the rest, and both should be learnt; but the integration formula is the one we are for the present most concerned with. It includes all the cases of integration that will be met with in the following chapters.

It is necessary, however, to know how to deal with the limits, as those figures placed at the top and bottom of the $\int$ are called.

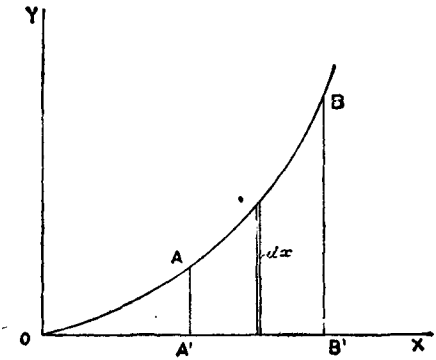

Fig. 2.

$\int_o^a x^n\,dx$ means, as has been already explained, that the addition or integration is carried through all values of x, from $x = o$ to $x = a$, and

$$\int_o^a x^n\,dx = \frac{1}{n+1}\,a^{n+1},$$

merely putting a in the place of x in the result. If it is required to add up or integrate from $x = b$ to $x = a$ the operation must be first performed from o to a and the integration from o to b subtracted.

Thus,
$$\int_b^a x^n\,dx = \frac{1}{n+1}\,a^{n+1} - \frac{1}{n+1}\,b^{n+1}$$
$$= \frac{1}{n+1}\,(a^{n+1} - b^{n+1}).$$

EXAMPLE.—Let OX and OY be two straight lines at right angles to each

other, and let AB be a curve, such that any point being taken in it, the perpendicular distance of this point from OY, squared and multiplied by c, will give its perpendicular distance from OX. This condition is expressed by the equation $y = c\,x^2$ where any point being taken in the curve, y is its perpendicular distance from OX and x its perpendicular distance from OY. It will be seen that although the particular values of x and y will be different for each point in the curve, the nature of the curve is such, that the relation $y = c\,x^2$ always holds good. From the points A and B in Fig. 2, draw AA^1 and BB^1 perpendicular to OX. It is required to find the area of AB B^1A^1.

Imagine a slit cut out of the area, parallel to OY and as thin as possible, so that we may denote its breadth by dx. Then since the length of it is y, its area will be $y\,dx$. But we may imagine the whole area to be made up of slits like this. The whole area will therefore be

$$\int_a^b y\,dx,$$

when a stands for OA^1 and b for OB^1, and x from being equal to a is supposed to grow gradually until it is equal to b. But $y = c\,x^2$,

Therefore, $$\int_a^b y\,dx = c\int_a^b x^2\,dx = \frac{c}{3}\,b^3 - \frac{c}{3}\,a^3.$$

I.—The Relation of Stress to Strain.

Stress means intensity of pressure, or pressure in proportion to area.

Strain means proportional alteration of length, or the ratio of the amount of the alteration to the original length.

Formula:— $$E = \frac{\frac{P}{S}}{\frac{a}{L}} = \frac{p}{i}$$

where L is the length of the piece of material strained; S its sectional area; P the force tending to extend or compress it; a the resulting alteration in length (increase or decrease); E the coefficient of longitudinal strain, also called Young's modulus of elasticity. p of course is equal to P/S, the stress, and $i = a/L$ the strain.

This formula constitutes Hook's law of elasticity; it is practically true of most materials for all values of p and i within certain limits

called the elastic limits of the material. These limiting values of p and i are called the proof stress and proof strain respectively. The formula is derived from experiment.

Corollary :— $$E = \frac{P\,L}{a\,S}$$

NOTE.—(i.) The piece of material must be prismatic, that is, straight and of uniform section.

(ii.) The force P must act through the axis of the prism, that is, through the centre of gravity of every section of it.

EXAMPLE :—(i.) AB, BC (Fig. 3) are two rods of wrought-iron 11 ft. and 7 ft. long respectively, their upper extremities are fixed at A and C respectively, and their lower extremities are welded together at B.

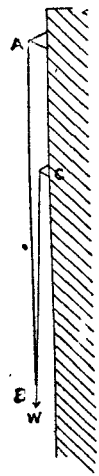

Fig. 3.

The sectional area of each rod being 3 square inches it is required to determine what weight may be safely attached to B. The value of E for wrought-iron is about 12,500 tons per square inch, and let us not permit a greater stress than 5 tons per square inch in either rod.

First consider the shorter rod C B.

From formula $$E = \frac{P\,L}{S\,a}$$

it follows that $$a = \frac{P\,L}{S\,E}.$$

Substituting the given values in this expression we obtain—

$$a = \frac{(3 \times 5)\ (7 \times 12)}{3 \times 12{,}500}$$

$$= \frac{84}{2{,}500} \text{ inches.}$$

Thus a, the absolute extension which is necessarily the same in both rods, must not be allowed to exceed $\frac{84}{2{,}500}$ inches.

Then considering the larger rod A B.

From formula $\quad E = \frac{P L}{S a}$; we get $P = \frac{E S a}{L}$.

Substituting $\quad P = \frac{12{,}500 \times 3 \times}{(11 \times 12)}\ \frac{84}{2{,}500}$

$$= \frac{105}{11} = 9\tfrac{6}{11} \text{ tons.}$$

The other rod we know sustains 5 tons per square inch, or 15 tons in all, so that the two rods together will support $24\frac{6}{11}$ tons.

EXAMPLE :—(ii.) Suppose the wall of a building has bulged, and in order to force it back it is necessary to exert a pressure of 40 tons. It is proposed to do this by means of the contracting force of a heated steel rod (modulus of elasticity about 13,000 per square inch). If the circumstances of the case permit of a rod 30 ft. long, what sectional area should it have ?

Suppose we make use of a range of temperature of 500 degs. Fahr. so that the laws of elasticity will still hold, and the value of E will not sensibly be affected ; then since steel expands ·000007 of its length for each degree the expansion of a 30 ft. rod will be—

$$\cdot 000007 \times 500 \times (30 \times 12)$$

$$= 1 \cdot 26 \text{ inches.}$$

Of this distance we may assume ·26 in. as likely to be absorbed in taking up slack, and of the inch remaining the sectional area of the rod may be such that $\frac{1}{3}$ in. is sufficient to bring about the pressure of 40 tons, so that the remaining $\frac{2}{3}$ in. can be devoted to producing movement in the brickwork to this amount.

From the formula $\quad E = \frac{P L}{S a}$; we get $S = \frac{P L}{a E}$.

Substituting $\quad S = \frac{40 \times (30 \times 12)}{13{,}000 \times \frac{1}{3}}$

$$= 3 \cdot 32 \text{ square inches.}$$

Thus the rod, if square, should measure $1\frac{7}{8}$ in. each way.

II.—The Resilience of Straight Bars.

Resilience is the work done in straining the bar up to its elastic limit

Formula :— $$R = \frac{V}{E} \cdot \frac{f^2}{2}$$

where V is the volume of the bar and R its resilience; E Young's modulus of elasticity, and f the proof stress of the material. Let S be the sectional area of the bar, and a the alteration in its length due to any stress P; also let $Q = fS$, and let P be used to denote any force less than Q.

Now, in the work of straining the bar, *i.e.*, of deforming it within elastic limits, the pressure brought to bear commences at nothing, and increases continuously up to Q, where it is supposed to stop. However rapidly or instantaneously this is done all the intermediate values of P must be passed through. Thus, in order to estimate the resilience or amount of work absorbed we must add up these different values of P multiplied by the little distances through which they act, which is to integrate $a\,dP$ from 0 to Q. In symbols :—

$$R = \int_0^Q a\,dP.$$

But from the formula $E = \frac{PL}{Sa}$ proved in the last chapter we know

$$a = \frac{LP}{ES};$$

Therefore, $$R = \int_0^Q \frac{LP}{ES}\,dP$$

$$= \frac{L}{ES}\int_0^Q P\,dP = \frac{L}{ES} \cdot \frac{Q^2}{2}$$

$$= \frac{L}{ES} \cdot \frac{f^2 S^2}{2} \text{ because } Q = fS.$$

And $LS = V$ the volume of the bar,

Therefore, $$R = \frac{V}{E} \cdot \frac{f^2}{2}.$$

Corollary :—From the formula

$E = \frac{PL}{aS}$ given in the last chapter we obtain $P = \frac{aSE}{L}$.

Let l be the alteration in length (increase or decrease) corresponding to the force Q,

then $$Q = \frac{lSE}{L}.$$

Now $$R = \frac{L}{E S} \cdot \frac{Q^2}{2}$$

has just been shown as the immediate result of integration.

Therefore, $$R = \frac{L}{E S} \cdot \frac{l S E}{L} \cdot \frac{Q}{2}$$

or $$R = l \frac{Q}{2}.$$

From this we gather, that a load equal to Q which strains a rod or bar up to its limit of elasticity, must be applied bit by bit; for if it were applied all at once, the whole load would act through the distance l, and the work done on the bar would be $l\,Q$ instead of $l \frac{Q}{2}$ which is as much as it can stand.

EXAMPLE.—A steel rod, one of the members of a machine, having become worn at one end is replaced by another having that half of its length of double sectional area, the other half remaining as before. How is this member of the machine affected in respect to its capacity of sustaining shocks?

Let $\frac{V}{E} \cdot \frac{f^2}{2}$ be the resilience of the original rod. The resilience of the smaller half of the new bar $= \frac{\frac{1}{2}V}{E} \cdot \frac{f^2}{2}$

and when this amount of work is performed on the smaller half, the amount performed on the larger

$$= \frac{V}{E} \cdot \frac{(\frac{1}{2}f)^2}{2}$$

for its section being twice as great, the intensity of pressure experienced there must be proportionally less. So the total amount of work that can be done on the new bar, without over straining the smaller one is

$$\frac{\frac{1}{2}V}{E} \cdot \frac{f^2}{2} + \frac{V}{E} \cdot \frac{(\frac{1}{2}f)^2}{2}$$

$$= \frac{V}{E} \cdot \frac{f^2}{2} (\tfrac{1}{2} + \tfrac{1}{4})$$

$$= \tfrac{3}{4} \frac{V}{E} \cdot \frac{f^2}{2}.$$

Thus the power to resist shock is diminished by $\frac{1}{4}$; and this is always the result of a partial increase of section.

III.—The Relation between Bending Moment and Shearing Force.

The *bending moment* on any section of a beam is the sum of the moments of all external forces acting on either part of the beam, about the centre of the section.

The *shearing force* on any section of a beam is the sum of all external forces acting on either part of the beam resolved parallel to the section, if not so already.

At the same time a *beam* is generally supposed to be subjected only to forces acting at right angles to it.

Formula :— $$M = \int_0^x F\,dx.$$

Where M is the bending moment on a section distance x from one end of the beam, and F is the shearing force on the same section.

The bending moment on any section of a beam is the sum of the moments of those very forces which constitute the shearing force. Hence, when these moments can be added up by integration, $M = \int F dx$.

For instance :—Let AB (Fig. 4) be a beam of span l uniformly loaded between the two supports with w lbs. per foot run. Then the shearing force on a section distance x from one end will be $\frac{wl}{2} - wx$.

By formula $M = \int_0^x F\,dx$, the bending moment on this section will be

$$\int_0^x \left(\frac{wl}{2} - wx\right) dx$$

$$= \frac{wl}{2}\int_0^x dx - w\int_0^x x\,dx$$

$$= \frac{wl}{2}\,x - w\,\frac{x^2}{2}.$$

As we proceed from the support at one end of the beam towards the centre, we find that the bending moment increases, though the shearing force does no such thing. Now, since the bending moment is the sum of the moments of the shearing forces, we can see that, at the point where the shearing force changes sign and begins to act in the opposite direction, the bending moment will begin to decrease instead of increasing as before. Hence, this point must mark the place where the bending moment reaches its greatest value; thus, the bending moment is greatest where the shearing force changes sign. But in order for the shearing force to change sign from + something to — something, or *vice versâ*,

it must pass through the value of nothing at all. Hence the rule, that where the shearing force is zero the bending movement is greatest.

EXAMPLE :—Let AB (Fig. 5) be a girder supporting a load which increases constantly, from nothing at A to w_o lbs. per foot-run at B, as indicated by the dotted line in Fig. 5 ; such a load as might be brought about by a drift of snow, or careless storing of material. The load per foot-run at distance x from end A of the girder is $\frac{x}{l}\, w_o$ lbs. per foot-run. And the shearing-force at this point will be

$$\frac{1}{3} \cdot \frac{w_o}{2}\, l - \frac{x}{l} \cdot \frac{w_o}{2}\, x.$$

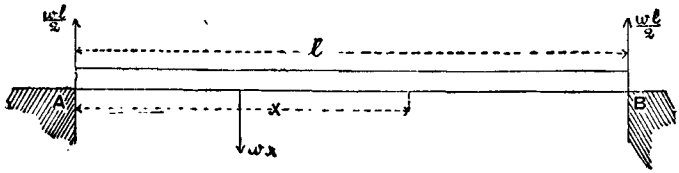

Fig. 4.

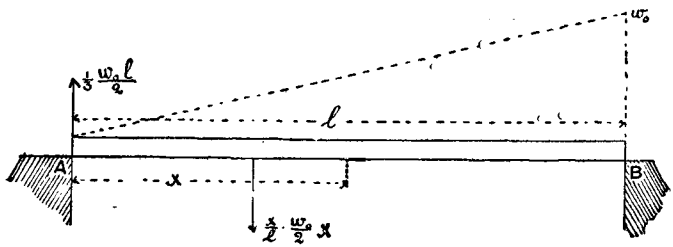

Fig. 5.

equating the shearing force to zero,

$$\frac{w_o}{6}\, l = \frac{x^2}{l}\, \frac{w_o}{2}.$$

Whence, $$x = \sqrt{\tfrac{1}{3}}\, l = .577\, l.$$

Hence, at distance $.577\ l$ feet from A, the shearing force is nothing, and it is here that the maximum bending moment will occur.

By formula $M = \int_o^x F\, dx$ the bending moment on a section distance x from end A is

$$\int_o^x \left(\frac{w_o}{6}\, l - \frac{x^2}{l}\, \frac{w_o}{2} \right) dx$$

$$= \frac{w_o}{6} \, l \int_o^x dx - \frac{w_o}{2l} \int_o^x x^2 d$$

$$= \frac{w_o}{6} \, lx - \frac{w_o}{2l} \frac{x^3}{3};$$

and the greatest bending moment is when $x = \cdot 577 \, l$. Therefore, substituting this value for x,

$$\text{maximum bending moment} = \frac{w_o}{6} \cdot 577 \, l^2 - \frac{w_o}{6} (\cdot 577)^3 \, l^2$$

$$= \frac{w_o}{6} \, l^2 (\cdot 385) = \cdot 067 \, w_o \, l^2$$

COROLLARY :—From the equation $M = \int F \, dx$, it will be seen at once, by referring to the introduction, that

$$\frac{dM}{dx} = F.$$

NOTE.—When the loading of a beam is irregular, so that a formula obtained for the shearing-force only applies to part of the beam, care must be taken not to allow for x a value which brings it outside the region to which the formula applies.

IV.—THE DISTRIBUTION OF STRESS ON THE CROSS SECTION OF A BEAM.

The *mean fibres* of a beam are those which are neither compressed nor extended when the beam is loaded and bent. The term *neutral axis* applies to the cross section of a beam, being the line where it is cut by the mean fibres.

Formula :— $$p = E \frac{x}{R},$$

where R is the radius of curvature of the mean fibres at the place where the cross section is taken, E is the modulus of elasticity of the material, and p is the stress or intensity of pressure at a point in the cross section distance x from its neutral axis.

Let Fig. 6 represent, in elevation, the part of the beam where the cross section is taken, before bending moment is applied, and let Fig. 7 represent the same part after loading has rendered it slightly curved. We know that on the inside of the curve Fig. 7 the fibres of the beam will be

in compression, and on the outside in tension ; also that somewhere between there will be fibres called the mean fibres of the beam, which are neither in compression or tension. These latter fibres cut the cross section at its neutral axis. Let AB (Fig. 6) be the section considered, and let CD be another section distance d from the former, d, however, being so small that the amount of the bending moment is virtually the same on both sections. Then, in Fig. 7, AB and CD will be slightly inclined to each other, and, being produced, must at length meet. Let them meet at O. Draw EF for the mean fibre of the beam between the two sections, and let MN be any other fibre distance x from EF. Then if R is the radius of curvature of the mean fibre, OE = OF = R. Also θ being the circular measure of the angle EOF, the line EF is equal to Rθ. But EF is one of the mean fibres of the beam, so that its length, not having been altered by bending, is still equal to d. Hence R $\theta = d$.

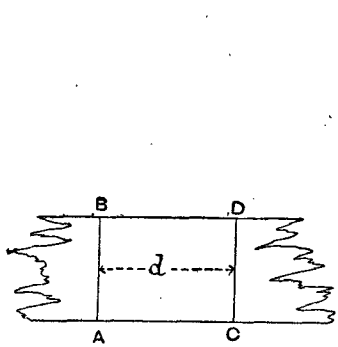

Fig. 6.

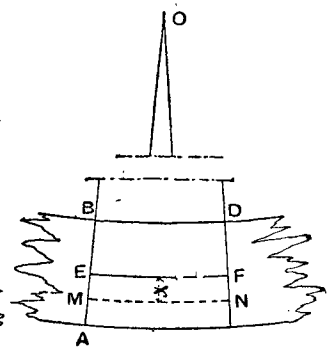

Fig. 7.

Now let us consider the fibre MN ; its length is now θ (R $\pm x$) ; plus if it is beyond EF, and minus if it is nearer to the inside of the curve. Also its original length was d ; therefore its alteration in length is θ (R $\pm x$) $- d$. And the strain upon it is—

$$\frac{\theta\,(\mathrm{R} \pm x) - d}{d} = \frac{\theta\,\mathrm{R} \pm \theta x - \theta\,\mathrm{R}}{d}$$

$$= \frac{\pm\,\theta x}{d} = \pm\,\frac{x}{\mathrm{R}}.$$

Also the stress upon it is E times the strain, as is seen in the formula

$$\mathrm{E} = \frac{p}{i};$$

therefore

$$p = \mathrm{E}\,\frac{\pm\,x}{\mathrm{R}}$$

the plus sign indicating tension, and the minus sign compression. Thus the intensity, if on the cross section of a beam, varies with the distance from its neutral axis.

It remains to show where this neutral axis is situated. Let Fig. 8 represent the section of the beam in question, and AB its neutral axis or

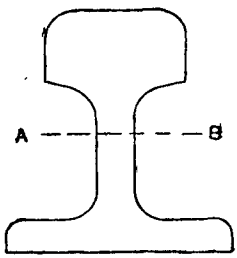

Fig. 8.

line where no stress occurs; all we know as yet is that this is somewhere between the top and bottom of the section. Imagine the whole section to be divided up into a number of small areas s each. The pressure on each of these would be $E \frac{\pm x}{R} s$.

Now, as the beam is supposed to be subjected to bending moment only (that is to say, since there are no forces at the ends of the beam tending either to compress or extend it as a whole, lengthways), the forces on the small areas of the section, when added together top and bottom, must come to nothing, because the section does not move. Hence the values of $E \frac{\pm x}{R} s$, when all of them are taken and added up, will amount to zero. Suppose now the cross section consisted of a stiff sheet of paper divided into small equal parts, the *weight* of each of which was s. Then $E \cdot \frac{x}{R} \cdot s$ would be the moment of the weight of one of these little divisions of the paper about the neutral axis, x being the distance of the division from the neutral axis. But the sum of these values has been shown to be necessarily equal to nothing. That is to say, the sum of the moments of these small bits of paper about the neutral axis comes to nothing.

Therefore the neutral axis must pass through the centre of gravity of the section. This is by definition of the centre of gravity.

N.B.—This formula is very important; nearly all those which follow are based upon it; but it has little *direct* practical application, since R

generally varies for different points in the length of the beam, and can seldom be readily measured.

COROLLARY:—If X is the greatest distance of the top or bottom edge of a section from its neutral axis,

$$p_o = E \frac{X}{R}$$

will be the greatest stress that occurs there.

EXAMPLE:—What is the diameter of the smallest circle into which a steel bar 1 in. × ¼ in. can be bent without introducing a stress greater than 7 tons per sq. in.?

From the formula $p_o = E \frac{X}{R}$ we obtain $R = E \frac{X}{p_o}$. Substituting, we get $R = 18{,}500 \frac{\frac{1}{8}}{7}$, where R is the radius of the circle, and 18,500 tons per sq. in. is the value of E; *i.e.*, R = 330 in., or 27½ ft Therefore the diameter of the required circle will be about 55 ft.

V.—THE MOMENT OF RESISTANCE OF A BEAM.

Formula :— $$M = \frac{p}{x} I,$$

where any cross section of the beam being taken, I is its moment of inertia and M the bending moment; also p in the intensity of pressure at a point in the section distance x from, its neutral axis.

Take any cross section of the beam, then either of the two parts into which the beam is divided may be regarded separately, as in equilibrium, the external forces being balanced by the stresses acting upon the cross section; *i.e.*, since both parts into which the beam is divided are subjected to the action of forces external to them and yet remain fixed, it is evident that these external forces must be held in equilibrium by the stresses on the cross section. Therefore if M is the moment produced by external forces on the section (the bending moment, that is), the sum of the moments produced by the small internal stresses upon it must be equal to M. Fig. 9 represents the cross-section of a beam, which may take any form; AB is drawn through its centre of gravity and represents the neutral axis. Let the area of the section be divided up into a great number of very thin slips parallel to AB; the length of these (which

varies) we will represent by y, and their breadth, since it is to be as small as possible, can be represented by dx. From the last chapter we know that the *intensity* of pressure upon these slips will be $\mathrm{E}\,\frac{x}{\mathrm{R}}$; hence the

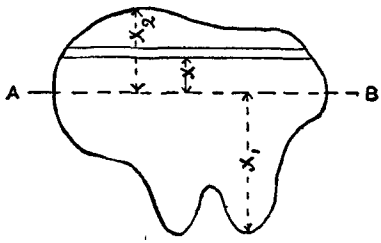

Fig. 9.

whole pressure upon each of them will be $\left(\mathrm{E}\,\frac{x}{\mathrm{R}}\right)(y\ dx)$; and the *moment* of this pressure about the neutral axis will be, for each of these little slips,

$$x\left(\mathrm{E}\,\frac{x}{\mathrm{R}}\,y\ dx\right) = \mathrm{E}\,\frac{x^2}{\mathrm{R}}\,y\,dx.$$

Now, if the bottom edge of the section is at a distance $-x_1$ from the neutral axis, and the quantity x be considered to grow from this value up to zero (which of course it reaches at the neutral axis), and then on to $+x_2$, which it attains at the upper extremity of the section; and if we add up all the values through which $\mathrm{E}\,\frac{x^2}{\mathrm{R}}\,y\,dx$ passes during this process, we shall have the whole moment produced by the stresses acting on the section; viz.,

$$\int_{x_1}^{x_2} \mathrm{E}\,\frac{x^2}{\mathrm{R}}\,y\,dx; \quad \text{or} \quad \frac{\mathrm{E}}{\mathrm{R}}\int_{x_1}^{x_2} x^2\,y\,dx;$$

since E and R are constants unaffected by the integration. This being the whole amount of the moment on the section produced by the internal stresses, it is equal to M the bending moment due to external forces; therefore,

$$\mathrm{M} = \frac{\mathrm{E}}{\mathrm{R}}\int_{x_1}^{x_2} x^2\,y\,dx.$$

The expression $\int_{x_1}^{x_2} x^2\,y\,dx$ is called the *moment of inertia* of the section, a most misleading appellation, as the idea of inertia is not wanted at all. The further elucidation of this integral is the purpose of the next chapter; in the meanwhile, representing it by the conventional symbol I, we have

$$M = \frac{E}{R} I \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(I)}.$$

But from the formula $p = E \frac{x}{R}$, shown in the last chapter, we obtain $R = E \frac{x}{p}$, and, substituting this value of R in the equation marked (I), we get

$$M = \frac{p}{x} I.$$

COROLLARY :—Giving x its greatest value X, the formula becomes

$$M = \frac{p_o}{X} I,$$

where p_o is the highest stress that obtains in the section.

EXAMPLE :—A rolled iron beam of height 10 in., whose flanges are of equal size, has a moment of inertia measured by 250 in inches. Find the greatest bending moment that should be permitted to act upon it.

Take $4\frac{1}{2}$ tons per sq. in. for the working stress. The flanges of the beam being equal in all respects, the centre of gravity of its section and therefore the neutral axis will be at half the height. So x cannot anywhere exceed 5 in. Hence, putting $p_o = 4\frac{1}{2}$ tons per sq. in., and X = 5 in. in the formula $M = \frac{p_o}{X} I$, we find that the moment of external forces should not exceed $\frac{4\frac{1}{2}}{5} 250 = 225$ inch tons, or 19 foot tons (nearly).

COROLLARY :—*Conditions for Circular Curvature.*—From the equation $M = \frac{E}{R} I$, marked (I) in the above, we obtain $R = \frac{E}{M} I$; of the quantities on the right-hand side of this equation, E, of course, is always constant, and when the form and area of cross section is the same throughout the length of the beam, I is also constant ; under these circumstances it will be seen that R varies inversely with M, the bending moment, so that where the latter is constant R also is constant, and the curvature becomes that of a circle.

Thus any straight piece of material (of the same section throughout its length) is bent into a circular arc when subjected to uniform bending moment.

Also—when a built girder is designed, so that M varies inversely with I under a full load, as is usually the case, $\frac{E}{R} I$ is constant, and R is therefore the same for all cross sections, thus making the curvature that of a circle throughout the span.

VI.—To find the Moment of Inertia of an Oblong Section.

Formula:— $$I = \frac{h^3 b}{12},$$

where h is the height, and b the breadth of the oblong.

As stated in the last chapter, *moment of inertia* is simply a name given to the integral

$$\int_{x_1}^{x_2} x^2 y \, dx;$$

x being always measured from the neutral axis unless otherwise stated.

If x is to be measured from any other line,

$$\int_{x_1}^{x_2} x^2 y \, dx$$

is described as the moment of inertia of the section about that particular line; but when moment of inertia is spoken of without further reference, it is always understood to mean the moment of inertia of the section about its neutral axis.

For a rectangular section, the centre of gravity being in the middle of the figure, the neutral axis will be at half the height, so that $x_1 = x_2 = \frac{h}{2}$. Also $y = b$, which is constant, and being unaffected by the integration, can be placed outside.

Therefore,
$$I = b \int_{-\frac{h}{2}}^{+\frac{h}{2}} x^2 \, dx$$
$$= b \left\{ \tfrac{1}{3} \left(\frac{h}{2}\right)^3 - \tfrac{1}{3} \left(\frac{-h}{2}\right)^3 \right\}$$
$$= \frac{h^3 b}{12}.$$

For example:—The moment of inertia of the section of a beam 7 in. × 5 in. is

$$\frac{7^3 \times 5}{12} = 143; \quad \text{or } \frac{5^3 \times 7}{12} = 73;$$

according to its position; the former when 7 in. is made the height of the section, the latter when 5 in. is.

Note.—The neutral axis must be parallel to one side of the oblong. The moment of inertia of a rectangular section, in ◇ position, is quite different.

VII.—To find the Moment of Inertia of a Symmetrical I Section.

Formula:— $I = \frac{1}{12}\left\{ h^3 b - (h - 2 t_f)\ (b - t_w) \right\}$,

where h is the height of the section, b its breadth; also t_w and t_f are the thicknesses of the web and the flanges respectively.

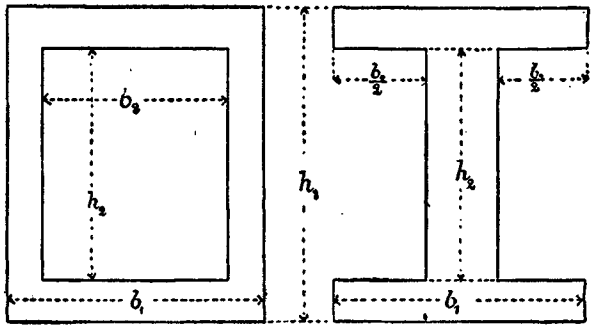

Fig. 10. Fig. 11.

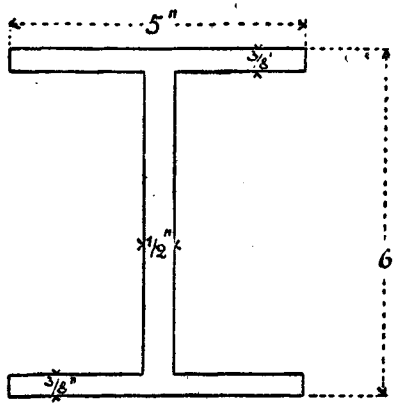

Fig. 12.

We have seen in the last chapter that the moment of inertia of an oblong section is

$$\frac{h^3 b}{12}$$

If the oblong is hollowed out symmetrically, as in Fig. 10, its external dimensions being h_1, b_1, and its internal h_2, b_2, the moment of inertia is evidently

$$\frac{h_1^3 b_1}{12} - \frac{h_2^3 b_2}{12},$$

and the result is the same when the parts of the section are arranged as in Fig. 11. In the latter case we can put $h_2 = h - 2\,t_f$ and $b_2 = b - t_w$. The moment of inertia then becomes

$$\tfrac{1}{12}\left\{ h^3 b - (h - 2\,t\ )^3\ (b - t_w) \right\}.$$

NOTE.—It must be observed that this formula is not applicable unless the section *is* symmetrical; that is, the top and bottom flanges must be equal in all respects, the assumption being that the neutral axis passes through the middle of the height. When the flanges or booms are unequal in area or disposition, the next chapter must be consulted.

EXAMPLE:—Find the moment of inertia of a rolled iron beam 6 in. × 5 in. (see Fig. 12); the mean thickness of web being $\frac{1}{2}$ in., and the mean thickness of flanges $\frac{3}{8}$ in.

By formula:— $\quad I = \tfrac{1}{12}\left\{ h^3 b - (h - 2\,t_f)^3\ (b - t_w) \right\}.$

Moment of inertia:— $= \tfrac{1}{12}\left\{ 216 \times 5 - (5\tfrac{1}{4})^3\ 4\tfrac{1}{2} \right\}$

$= 35{\cdot}74$ in inches.

VIII.—MOMENT OF INERTIA ABOUT ANY AXIS.

Formula:— $\quad I_m = I_n + D^2 S,$

where S is the area of the section; I_n the ordinary moment of inertia about its neutral axis; I_m its moment of inertia about another line distance D from the neutral axis, and parallel to it.

Let the Fig. 13 represent a cross section of any form, of which N N is the neutral axis, and M M any other line parallel to it, and distance D from it.

Then, $\quad I_m = \int x_1^2\, y\, dx$; x_1 being measured from M M

$= \int (x + D)^2\, y\, dx$; x being measured from N N,

for $x_1 = x + D$ (see Fig. 13).

Therefore, $I_m = \int (x^2 + 2\,x\,D + D^2)\,y\,dx$

$$= \int x^2\,y\,dx + 2\,D \int x\,y\,dx + D^2 \int y\,dx.$$

The first term of this expression is equal to I_n, being the ordinary moment of inertia of the section about its own neutral axis. Of the second term, the integral part is the moment of area of the section about its neutral axis; but the neutral axis passes through the centre of gravity of the section, and the moment of area of a figure about a line passing through its centre of gravity is zero; this is by definition of centre of gravity. So the second term is nothing, because

$$\int x\,y\,dx = 0.$$

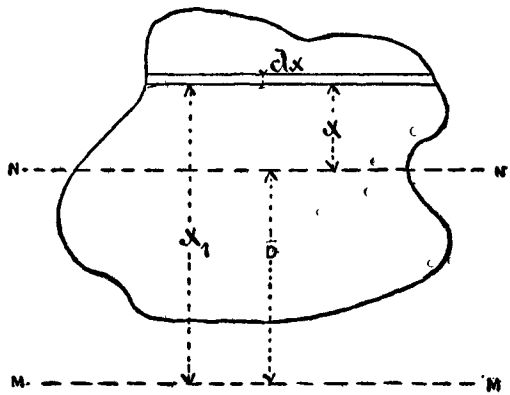

Fig. 13.

With regard to the third and last term, $\int y\,dx$ is the area of the section, so this term is equal to D^2 S. Hence we have

$$I_m = I_n + D^2\,S.$$

EXAMPLE:—(i.) An old piece of bar iron, with a longitudinal line of rivet holes in it, is to be used for a temporary structure in a position where it will be subjected to bending moment.

Fig. 14 shows the cross section through the rivet holes. What moment of inertia may be counted upon?

If the piece is presented with its flat surface to the forces tending to bend it, the neutral axis is C D passing through the middle of the section; and the moment of inertia is found at once by the formula,

$$I = \frac{h^3 b}{12} \text{ (see Chapter VI.),}$$

being first applied to the whole section as if complete, and then to the area of the rivet hole, in order to subtract the amount due to the latter.

Thus, moment of inertia $= \frac{(\frac{3}{4})^3 4}{12} - \frac{(\frac{3}{4})^3 \frac{3}{4}}{12}$

$$= \frac{27}{64 \times 12}(4 - \tfrac{3}{4}) = \cdot 114 \text{ in inches.}$$

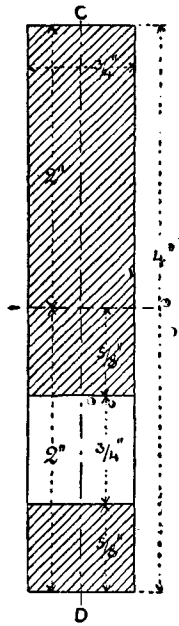

Fig. 14.

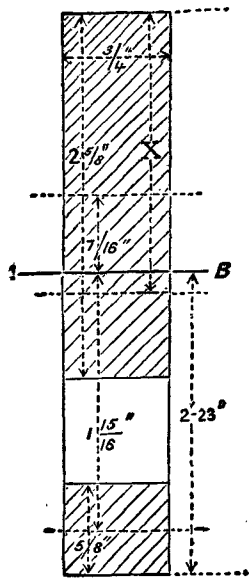

Fig. 15.

If, on the other hand, the piece presents its edge to the forces, tending to bend it, we must first find the position of the neutral axis, for it will not be at the middle of the height.

Let X be the distance of the centre of gravity of the section from the top of it, as seen in Fig. 15.

Then, taking moments of area,

$$X\,(\tfrac{3}{4} \times 4 - \tfrac{3}{4} \times \tfrac{3}{4}) = 2\,(\tfrac{3}{4} \times 2) - 1 \times (\tfrac{3}{4})^2;$$

$$X\,(3 - \tfrac{9}{16}) = 6 - \tfrac{9}{16};$$

$$X = \tfrac{87}{39} = 2{\cdot}23 \text{ inches.}$$

Accordingly, drawing A B for the neutral axis 2·23 in. above the bottom of section, Fig. 15, we find it $\frac{7}{16}$ in. distant from the centre of the larger part of the section, and $1\frac{15}{16}$ in. distant from the centre of the smaller.

The formula, $I_m = I_n + D^2 S$,

can now be applied to find the moment of inertia of either part of the section about A B, the neutral axis of the whole.

For the larger part of the section, moment of inertia about A B

$$= \frac{(2\frac{5}{8})^3 \frac{3}{4}}{12} + (2\frac{5}{8} \times \frac{3}{4}) (\frac{7}{16})^2 = 1{\cdot}125 + {\cdot}378 = 1{\cdot}5.$$

For the smaller part of the section, moment of inertia about A B

$$= \frac{(\frac{5}{8})^3 \frac{3}{4}}{12} + (\frac{5}{8} \times \frac{3}{4}) (1\frac{15}{16})^2 = {\cdot}015 + 1{\cdot}75 = 1{\cdot}765.$$

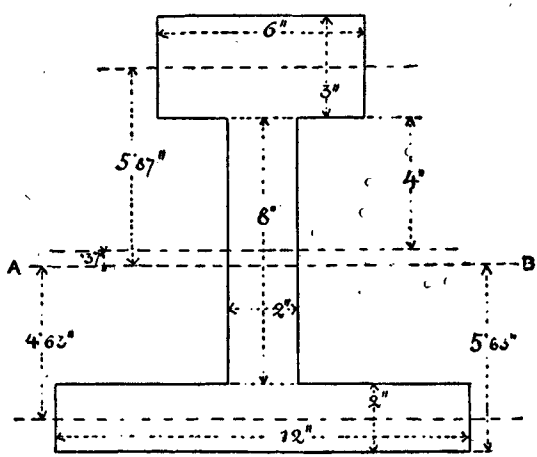

Fig. 16.

Adding these together, we obtain 3·71, the moment of inertia of the whole section about its neutral axis.

EXAMPLE:—(ii.) To find the moment of inertia of the section shown in Fig. 16. First, to obtain the position of the neutral axis.

Let X be the distance of the centre of gravity of the section from the bottom of the lower flange.

Then, taking moments of area,

$$X\,(12 \times 2 + 2 \times 8 + 6 \times 3)$$
$$= (12 \times 2)\,1 + (8 \times 2)\,(4 + 2) + (6 \times 3)\,(8 + 2 + 1\tfrac{1}{2}),$$

whence $X = 5{\cdot}63$ inches.

Now, the moment of inertia of the whole section about its proper neutral axis A B, as just found, will be made up of the moment of inertia of the upper flange about A B, the moment of inertia of the web about A B, and the moment of inertia of the lower flange about it.

By formula, $I_m = I_n + D^2 S$.

The moment of inertia of the upper flange about A B

$$= \frac{3^3 \times 6}{12} + (6 \times 3)\ (5{\cdot}87)^2 = 13{\cdot}5 + 620{\cdot}2 = 633{\cdot}7.$$

Also, the moment of inertia of the web about A B

$$= \frac{8^3 \times 2}{12} + (8 \times 2)\ (\cdot 37)^2 = 85{\cdot}33 + 2{\cdot}192 = 87{\cdot}5.$$

Similarly, the moment of inertia of the lower flange about A B,

$$\frac{2^3 \times 12}{12} + (12 \times 2)\ (4{\cdot}63)^2 = 8 + 514{\cdot}3.$$

Therefore, the moment of inertia of the whole section about A B, its neutral axis, is

$$633{\cdot}7 + 87{\cdot}5 + 522{\cdot}3 = 1243{\cdot}5 \text{ in inches.}$$

NOTE.—In calculating the moment of inertia of large I sections, that of the web is often left out of account, for additional safety. Also, in applying the formula, $I_m = I_n + D^2 S$, to the booms, I_n is generally neglected as small in comparison to $D^2 S$. If, further, the two booms are equal in all respects, the neutral axis comes at the middle of the height, so that $D = \frac{H}{2}$ and $D^2 S = \left(\frac{H}{2}\right)^2 S$, being the same for both booms. Thus, the whole moment of inertia of a large section becomes $\frac{1}{2} H^2 S$, where H is roughly the height of the girder.

The moment of inertia of a circular section is $\frac{\pi}{4} R^4$. This must be taken as granted for the present; it will be proved later on, when we come to the subject of torsion.

IX.—RADIUS OF GYRATION FORMULA.

Formula:— $$k = \sqrt{\frac{I}{S}},$$

where I is the moment of inertia of a section whose area is S, and k is its radius of gyration.

The expression "radius of gyration," like "moment of inertia," is an ill-advised term, perverted from the science of dynamics. In this subject, at least, it is only necessary to define it as equal to

$$\sqrt{\frac{I}{S}},$$

remembering that it is measured from the same line about which I is taken, which, unless otherwise stated, is always the neutral axis of the section.

For example:—To find the radius of gyration of an oblong section. Let the dimensions of the section be h and b.

Then the moment of inertia is

$$\frac{h^3\,b}{12} \text{ (see Chapter VII).}$$

Hence, by the above formula,

$$k = \sqrt{\frac{\frac{h^3\,b}{12}}{h\,b}} = \frac{h}{\sqrt{12}},$$

the radius of gyration being thus independent of b.

Again:—To find the radius of gyration of a circular section. The moment of inertia about a diameter is $\frac{\pi}{4} R^4$, as stated at the end of the last chapter. Therefore, by the above formula,

$$k = \sqrt{\frac{\frac{\pi}{4} R^4}{\pi R^2}} = \frac{R}{2},$$

which is measured from the centre, at right angles to the diameter about which the moment of inertia is taken.

X.—The Distribution of Stress upon a Section of any Structure, when it is acted upon by Bending Moment, in combination with Forces which produce Uniform Normal Stress.

Formula :— $$p = p_u \pm \frac{M}{I} x,$$

where p_u is the uniform stress referred to; M the moment acting in combination with it; I the moment of inertia of the section; and p the intensity of pressure at a point in the section distance x from a line

in the same, which passes through its centre of gravity, and is at right angles to the direction in which bending tends to take place.

This result is obtained by treating the structure (whatever it may be) as if it were a beam; we already know the distribution of stress produced by a bending moment on the cross section of a beam (see Chapters IV. and V.), to this the uniform stress has simply to be added or subtracted, as the case may be.

First, then, treating the structure as a beam, and considering bending moment only, the stress due to M at a point in the section distance x from its neutral axis is $\frac{M}{I}x$. This is obtained from the formula $M = \frac{P}{x}I$, proved in Chapter V. In addition there is a uniform stress on the section, such as might be produced by the weight of material

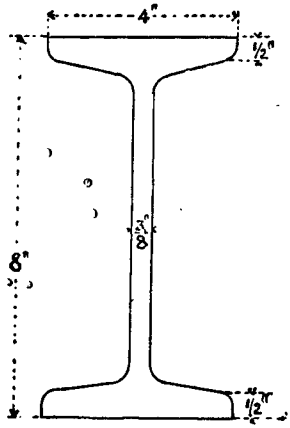

Fig. 17.

above it, or by some force acting through the centre of gravity of the section. This force is, of course, the same everywhere, and has been denoted by p_u.

p_u, then, must be added to or subtracted from $\frac{Mx}{I}$ according as they are both in the same sense or not. Thus, if they are both compressive or both tensile they will be added, and if one is compressive and the other tensile they must be subtracted.

Hence the total intensity of pressure at a point in the section distance x from the neutral axis is $p_u \pm \frac{M}{I}x$; *i.e.*, in correct language, $p_u \pm \frac{M}{I}x$ is the stress at a point distance x from a line passing through the centre

of gravity of the section at right angles to the direction in which bending tends to take place. For *neutral axis* means a line of no stress, and the addition of a uniform normal stress on the section makes the position of such a line entirely different from what it was at first, when the effect of bending moment alone was considered.

COROLLARY :—The greatest values of p, positive or negative, are obtained when x is given its extreme value,

$$p_o = p_u \pm \frac{M}{I} X.$$

EXAMPLE :—(i.) A rolled iron beam, spanning an opening of 12 ft. under a uniform load of 75 lbs. per foot run, is found to receive longitudinally the thrust of a brick arch, the amount of which is estimated at 12 tons. The dimensions of beam, in section, are given by Fig. 17. Let us discuss the safety of the beam.

The area of the section is :—

$$(4 + 4)\ \tfrac{1}{2} + (8\text{—}1)\ \tfrac{3}{8} = 6{\cdot}62 \text{ sq. in.},$$

therefore the thrust of the arch produces a uniform stress of $\frac{12}{6{\cdot}62}$, or 1·81 tons per sq. in.

Again, by the formula $I = \frac{1}{12}\left\{ h^3 b - (h - 2tf)^3 (h - tb) \right\}$,

the moment of inertia $= \frac{1}{12}\left\{ 8^3 \times 4 - (8 - 1)^3 (4 - \frac{3}{8}) \right\}$

(See Chapter VII.)

$= \frac{1}{12}(2048 - 343 \times \frac{29}{8})$,

$= 67$ in inches.

Also the bending moment which obtains at the centre of span :—

$$= \frac{75\ (12)^2}{8} \text{ foot lbs. or } \frac{75 \times (144)^2}{8 \times 2240} \text{ inch tons} = 86{\cdot}8.$$

So, applying the formula $p_o = p_u \pm \frac{M}{I} X$ to a cross section of the beam at centre of span, and taking the plus sign to obtain the added effect of the bending moment and longitudinal thrust,

$$p_o = 1{\cdot}81 + \frac{86{\cdot}8}{67} . 4 = 6{\cdot}99 \text{ tons per sq. in.}$$

Hence the upper flange of the beam is subject to a maximum stress of 7 tons per sq. in.

EXAMPLE :—(ii.) A commanding position has been chosen for the site of a stone memorial column, which is to be built of uniform circular section throughout to the height of 100 ft. above the plinth. Taking the maximum wind pressure at 56 lbs. per sq. ft., what is the least diameter that should be given to the column ?

Let the necessary diameter be 2 R feet.

Such a column, standing by itself in still air, would be in simple compression due to its own weight; the effect of the wind is to bend it like a cantilever, so as to increase this pressure on the lee side, and diminish it on the windward; only we must provide that it is nowhere diminished to zero, or rather not beyond zero (as after this the joints of the structure would tend to open). This condition is satisfied when in the formula $p_o = p_u \pm \frac{M}{I} X$ (the minus sign being taken) $p_o = 0$;

i.e., when $p_u = \frac{M}{I} X$, (i.).

Let us consider a cross section taken just above the plinth, where the moment due to wind pressure is greatest. The material of the column may weigh 1 cwt. per cubic ft., then a height of 100 ft. will create a uniform stress of 100 cwt. per sq. ft. on this section. So $p_u = 100$ cwt. per sq. ft.

The total pressure of wind on a cylinder is generally taken to he half that which would be exerted on a plane surface of the same diametrical section. Hence the total maximum pressure will in this instance amount to $\frac{1}{2}$ (100 × 2 R 56) lbs., or 50 R cwt., creating a moment of 50 R $\frac{100}{2}$ foot cwt.

Therefore $M = 2500$ R foot cwt.

Also, $I = \frac{\pi}{4} R^4$ (as stated at the end of Chapter VII.).

Substituting these values in the above equation,

$$100 = \frac{2500\ R}{\frac{\pi}{4} R^4} R$$

$$i.e.,\ \pi\ R^2 = 100,\ \text{or}\ R = \sqrt{\frac{100}{3{\cdot}14}} = 5{\cdot}6;$$

the diameter, therefore, should be not less than 11 ft.

NOTE.—Just as in Example i. the line of action of the thrust of the arch is supposed to pass through the centre of all cross sections of the beam, the slight effect of deflection in disturbing this being neglected; so in Example ii. the force due to the weight of the column is supposed to pass through the centre of the section, the slight effect of deflection in making it eccentric being neglected.

XI.—THE DISTRIBUTION OF STRESS ON A SECTION OF ANY STRUCTURE, WHEN IT IS ACTED UPON BY A NORMAL FORCE, THE LINE OF ACTION OF WHICH DOES NOT PASS THROUGH THE CENTRE OF GRAVITY OF THE SECTION.

Formula:— $$p = \frac{P}{S}\left\{1 \pm \frac{vx}{k^2}\right\}$$

where P is the normal force in question, v the distance of the point where its line of action cuts the plane of the section, from the centre of gravity of the same. S is the area of the section; k its radius of gyration; p the total intensity of pressure at a point distance x from a line drawn through the centre of gravity of the section at right angles to the direction of v.

Let AB, Fig. 18, be the section in question, and O its centre of gravity, also let the force P act at a point O^1 distance v from O. Now imagine two equal and opposite forces to be applied to the section at its centre of gravity; let them be both equal to P and like P act perpendicularly to the plane of the section. Then these two forces being equal and opposite will not affect the condition of things at all; so there is no harm in supposing them to actually exist, and we may henceforth regard the section as acted upon by a force P at its centre of gravity, (thus producing uniform stress) in combination with a couple, the moment of which is Pv. This moment tends to bend the structure (in a direction at right angles to CD) towards the side on which the original force P is situate, and produces the same distribution of pressure on the section as any other bending moment.

We can now proceed as in the last chapter. The stress produced by a bending moment at a point in the section distance x from the neutral axis is $\frac{M}{I}x$; this is obtained from the formula $M = \frac{p}{x}I$ proved in Chapter V. So the stress produced by the moment Pv at a point distance x from CD is $\frac{Pv}{I}x$. (For the line CD is drawn through the centre of gravity of the section, and thus coincides with the position of the neutral axis, or line of no stress, when the effect of the moment is considered alone.) Also the uniform stress due to the force P is $\frac{P}{S}$. Therefore the whole stress at any point in the section distance x from CD is

$$\frac{P}{S} \pm \frac{Pv}{I}x,$$

the plus or minus sign being taken according to which side of the section is considered, *i.e.*, according to whether the stress due to the moment is

of the same kind or of opposite kind to $\frac{P}{S}$ (for the force P may act in either sense, though in Fig. 18 it is a compressive force).

But $I = k^2 S$ from the formula $k = \sqrt{\frac{I}{S}}$. See Chapter IX.

Therefore $$p = \frac{P}{S}\left\{1 \pm \frac{vx}{k^2}\right\}$$

k of course being taken about the line CD in the same manner as x; see again Chapter IX.

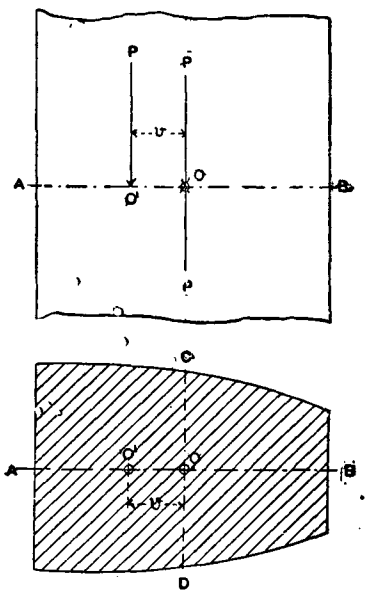

Fig. 18.

COROLLARY:—p_o the greatest value of p is obtained when x is given its extreme value and the plus sign is taken

$$p_o = \frac{P}{S}\left(1 + \frac{vx}{k^2}\right).$$

NOTE:—(i.) The section should be, and in practical cases always is, symmetrical about the line OO^1 in which v is measured.

(ii.) When the force P does not act normally to the section its resolved force in this direction may be taken.

EXAMPLE:—(i.) A tie has been formed of two pieces of timber

scantling 7 in. by 3 in., fastened with bolts as shown in Fig. 19. Find the strength of the tie, assuming one ton per square inch for working stress.

Considering either of the sections AB or CD, it will be seen that the force producing tension is applied at the edge of the section. Therefore in the formula $p_o = \frac{P}{S}\left(1 + \frac{vx}{k}\right)$; $v = x = 1\frac{1}{2}$ in.

also k^2 for a rectangular section equals $\frac{h^2}{12}$ (See Chapter IX.); h being the depth of the section in the direction in which bending tends to take place, 3 in. in this case.

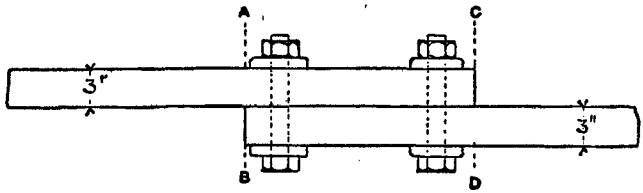

Fig. 19.

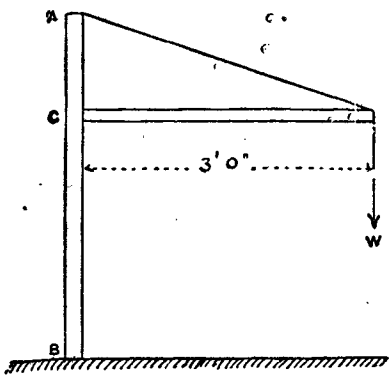

Fig. 20.

Substituting $$1 = \frac{P}{7 \times 3}\left(1 + \frac{1\frac{1}{2} \times 1\frac{1}{2}}{\frac{3^2}{12}}\right)$$

$$= \frac{P}{21}\left(1 + \frac{12}{4}\right).$$

Whence $P = \frac{21}{4}$ tons; just a quarter of what the tensional force on

the tie might have been, if the line of action of the force had been made to pass through the centre of all cross sections, as in a butt joint.

EXAMPLE :—(ii.) The letters ACB in Fig. 20 indicate a round post 6 in. in diameter, firmly fixed in the ground ; an arm is provided at C which is 3 ft. long. It is required to determine what weight may be appended to the end of this arm without endangering the post.

Considering any cross section of the post between B and C, it may be regarded as acted upon by a vertical normal force which does not pass through its centre of gravity, but cuts the plane of the section at a distance of 3 ft.

Hence applying the formula

$$p_o = \frac{P}{S}\left(1 + \frac{v\,x}{k^2}\right)$$

we substitute 36 in. for v and 3 in. for x.

Whence $$p_o = \frac{W}{\pi\ 3^2}\left(1 + \frac{36 \times 3}{(\frac{3}{2})^2}\right);$$

where W is the weight required and $\frac{3}{2}$ is the value of k. See Chapter IX.

Therefore $$W = \frac{p_o\ \pi\ 3^2}{1 + \frac{36 \times 3}{(\frac{3}{2})^2}}$$

Then if we make p_o equal to 18 cwt. per square inch,

$$W = \frac{18.\ \frac{7}{22}\ 9}{1 + 48} = 1 \text{ cwt. nearly.}$$

N.B.—When the line of action of the force P is not normal to the section in question, it is only necessary to substitute for P its resolved force in this direction, at the point where the line of action cuts the plane of the section.

XII.—THE STABILITY OF WALLS AND DAMS.

Formula :— $$v = \tfrac{1}{6}\ T$$

where, when T is the thickness of the wall or dam at the place where a section is taken, the line of action of the resultant of all forces acting on the section, cuts it at a distance v from the middle of the thickness of the wall.

When the conditions of Chapters X. and XI. apply simultaneously to

rectangular cross sections of masonry, as is usually the case with dams and retaining walls, etc., stability is best verified graphically by help of the above formula.

Fig. 21 shows a vertical section of such a structure (it may be conveniently taken to be one foot thick perpendicularly to the paper); AB being the trace of the section to be considered. Bisect AB at D. Let C be the centre of gravity of the part above AB and let EF be the line of action of the force acting on it (be it the pressure of earth or water, or that of a brickwork arch). Through C draw a vertical line to meet EF at E, and complete the parallelogram in the usual way, so that the line of action of the resultant force R from E cuts AB at G.

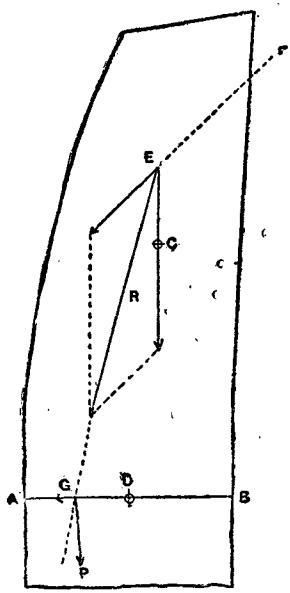

Fig. 21.

The formula states that DG should be equal to $\frac{1}{6}$ AB, a requirement which has nothing to do with the *magnitude* of R the resultant force just found. For let R be resolved at the point G into two forces, one parallel and the other at right angles to AB and let P measure the latter component.

The case is now reduced to that of the last chapter, the section AB being acted upon by a normal force P acting at G distance v from D. We may therefore apply the formula

$$p = \frac{P}{S}\left(1 \pm \frac{vx}{k^2}\right),$$

when the negative sign is taken and x receives its full value $\frac{1}{2}$ T; p should be zero; for if it is less, *i.e.*, of minus sign, it indicates that there is tensional stress at B tending to open the joints of the structure; and if it is much more, *i.e.*, a positive quantity, the weight of the superstructure can generally be reduced with economy.

Hence putting $p = 0$

$$1 = \frac{v\frac{1}{2}T}{k^2}; \text{ where } k^2 = \frac{T^2}{12}. \text{ See Chapter IX.}$$

Therefore $v = \frac{1}{6}$ T. Thus in Fig. 21, the chief condition of stability is satisfied when DG $= \frac{1}{6}$ AB. It should be remembered however that this is not the only condition. The component of R parallel to AB produces a shear, also the increased pressure at A, found by taking the plus sign of the above formula, must not exceed a reasonable amount.

NOTE.—The point G in AB is called the *centre of pressure* of the section; the line joining the centres of pressure of a series of sections is called the *line of resistance.*

XIII.—TO CUT THE STRONGEST RECTANGULAR BEAM OUT OF A STRAIGHT ROUND LOG.

Formula :— $$b = \cdot 57\ D,$$

i.e., b the breadth of the beam should be rather more than half D, the diameter of the log.

The strongest beam that can be cut from a log is the one which will have the greatest moment of resistance.

Now M, the moment of resistance, is equal to

$$\frac{p_o}{\overline{X}}\ I \text{ (see Ch. V., Coll.),}$$

of which I the moment of inertia is $\frac{h^3 b}{12}$; since the section of the beam is to be rectangular (see Ch. VI.).

Therefore $$M = \frac{p_o}{\frac{h}{2}} \cdot \frac{h^3 b}{12} = \frac{p_o}{6}\ h^2 b.$$

But $h^2 = D^2 - b^2$; because the diagonal of the beam will be a diameter of the log.

Thus $$M = \frac{p_o}{6} (D^2 - b^2) b.$$

First of all consider b the breadth of the proposed beam, to be taken very small as in Fig. 22. Then M will evidently also be small, and

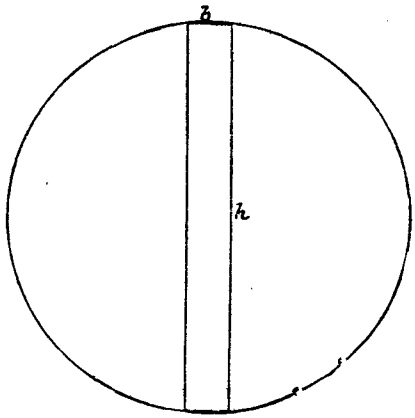

Fig. 22.

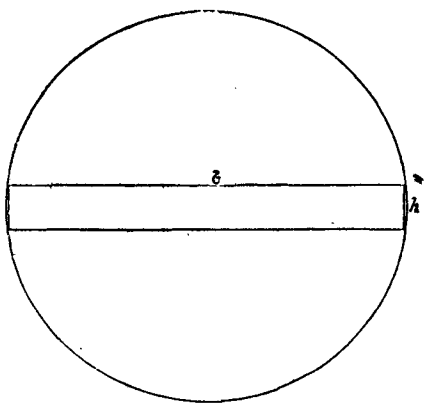

Fig. 23.

any little increase of b will make M larger. Now suppose b to be taken very large, nearly equal to D, as in Fig. 23; M however will be smaller than ever, and any further increase of b will make it smaller

still. From this it appears,—that as b is gradually increased from a small value to a large one, M is also increased, up to a certain point, after which it begins to diminish.

What we want to find, is the value of b at which M ceases to increase.

Again suppose b small,—let it be increased by db, so that M is also increased by the corresponding amount we call dM.

Then $\frac{d\mathrm{M}}{db}$ will be something, we don't know what it is, it may be anything, but not nothing.

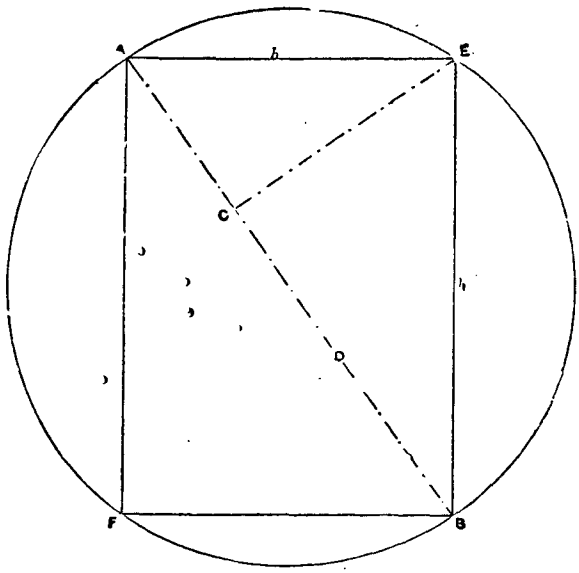

Fig. 24.

Now let b be imagined to grow until a further small increase does not increase M any more, so that $\frac{d\mathrm{M}}{db}$ is nothing, (dM being nothing).

This is the point we want to find, as after that M will begin to decrease until it approaches the state of things in Fig. 23.

Hence let us put $\frac{d\mathrm{M}}{db} = 0$ and see what happens—

$$\mathrm{M} = \frac{p_o}{6} (\mathrm{D}^2 - b^2)\, b\,;$$

as has been shown.

So $\frac{dM}{db}$; the differential coefficient of this expression is to be equal to zero.

Now the differential coefficient of

$$\frac{p_o}{6} D^2 b, \text{ is } \frac{p_o}{6} D^2,$$

and the differential coefficient of

$$-\frac{p_o}{6} b^3, \text{ is } -\frac{p_o}{6} 3 b^2.$$

Therefore $$\frac{p_o}{6} D^2 - 3 \frac{p_o}{6} b^2 = 0,$$

whence $$D^2 = 3 b^2$$

or $$b = \cdot 57 D,$$

this therefore is the best value for b.

Also since $h = \sqrt{D^2 - b^2} = \sqrt{3 b^2 - b^2} = b \sqrt{2}$,

the ratio of h to b is about $1\frac{1}{2}$ to 1

COROLLARY :—When it is desired to mark out the section on the log itself, in an accurate manner, the following construction may be adopted.

Let AB (Fig. 24) be the diameter of the log; take AC a third of AB, and draw CE at right angles to it.

Then AEB being the angle in a semicifcle is a right angle Therefore complete the rectangle AEBF, which will be the section required.

For the triangles AEB and ACE are similar.

Therefore $$\frac{AE}{AB} = \frac{AC}{AE}$$

or $$\frac{b}{D} = \frac{\frac{1}{3} D}{b}$$

Whence $b^2 = \frac{1}{3} D^2$; which is the condition required.

XIV.—THE NUMBER OF TURNS WHICH MAY BE MADE WITH SAFETY IN WINDING UP A SPIRAL SPRING.

Formula:— $$N = \frac{p_1 L}{\pi E t}.$$

Where L and t are the length and thickness of the spring respectively

E is the modulus of elasticity of the material, p_1 the greatest stress to be permitted in it, and N the number of turns (*i.e.*, complete revolutions) required.

Let a straight spring of length L be subjected to uniform bending moment and thus bent in to the form of the circular arc AB (Fig. 25). See Ch. V., Corollary.

From C the centre, draw CA, CB to the points A and B. Also from A and B draw tangents to the curve meeting at D, and produce AD to B^1.

Then regarding AB^1 as the original position of the spring, we see that it has been bent through the angle BDB^1, which is equal to the angle ACB, of which the circular measure is $\frac{L}{R}$.

Thus the spring has so far been bent through an angle $\frac{L}{R}$; put this equal to θ

then $$R = \frac{L}{\theta}.$$

But from the formula $p^o = E\frac{X}{R}$ (see Ch. IV.)

we obtain $R = E\frac{X}{p_o}$, or $R = E\frac{t}{2p_o}$, where t is the thickness of the spring and p_o the greatest stress that obtains in it.

Hence $$\frac{L}{\theta} = E\frac{t}{2p^o}.$$

But θ, instead of being a comparatively small angle, as in the figure, may be made a very large one and consist of several turns.

So let $\theta = 2N\pi$, where N is any number or fraction, and 2π of course represents 360°, *i.e.* a complete turn.

Then $$\frac{L}{2N\pi} = E\frac{t}{2p_o}.$$

Therefore $$N = \frac{p_o L}{\pi E t}.$$

And when p_o is made to be the working intensity of stress, here denoted by p_1, N becomes the greatest number of turns that should be allowed.

This result, proved in the case of a spring originally straight, will not be materially affected in accuracy in the case of a spring originally curved, and so it comes to hold very approximately for a spiral spring, N being the number of complete turns which may be given at the arbor.

NOTE.—It will be observed that neither the breadth of the spring, or the form of its section is of any account in the calculation, thus if the spring is of circular section its diameter d takes the place of t in the formula, which is otherwise unaffected.

EXAMPLE:—It is desired to design a spring to admit of six turns at the arbor. What should be the ratio of its length to its thickness?

We will suppose a steel spring, and allow a working stress of 18 tons per square inch.

Modulus of elasticity 12,000 tons per square inch.

Then from the formula $N = \frac{p_1 L}{\pi E t}$

$$\frac{L}{t} = \frac{\pi NE}{p_1}$$

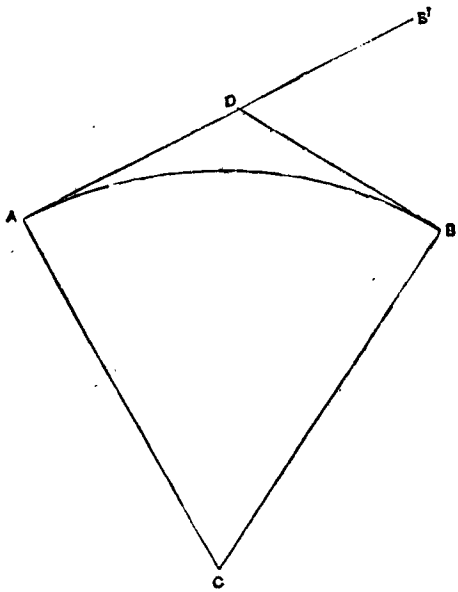

Fig. 25.

Therefore ratio required $= \frac{3{\cdot}1416 \times 6 \times 12{,}000}{18}$

$= 12{,}566$

NOTE:—The working stress used in designing small springs is generally very high indeed.

XV.—To Find the Moment exerted by a Spiral Spring after N turns of the Axle or Arbor have been made.

Formula:— $$M = 2\pi N \frac{E I}{L};$$

Where L and I are the length of the spring and the moment of inertia of its section respectively; E is the moment of elasticity; and M the turning moment created by N revolutions of the arbor.

As in the last chapter, consider an originally straight spring of length L to be bent into the circular arc AB (Fig. 25).

From C the centre draw CACB to the points A and B. Also from A and B draw tangents, meeting at D.

Produce AD to B^1.

Then regarding AB^1 as the original position of the spring, it will be seen that it has been bent through the angle BDB^1, which is equal to the angle ACB, the circular measure of which is $\frac{L}{R}$ where R is the radius of the arc.

But since $\frac{L}{R}$ may be an angle of any magnitude, let it be equal to $2N\pi$ where N is any number or fraction. Then $R = \frac{L}{2N\pi}$.

From the formula $M = \frac{E}{R} I$; proved incidentally in Chapter V.

$$R = \frac{E}{M} I.$$

Therefore $$\frac{L}{2N\pi} = \frac{E}{M} I.$$

Whence $$M = 2\pi N \frac{EI}{L},$$

which result will not be materially affected if the spring instead of being originally straight, be originally curved, in manner a spiral spring is.

Example:—A spiral spring, 60 feet long, $\frac{1}{4}$ in. thick, and 1 in. broad, is given one and a-half turns, what moment does it then exert?

The moment of inertia is in this case

$$\frac{(\frac{1}{4})^3 \times 1}{12};$$

this is obtained from the formula $\frac{h^3 b}{12}$ proved in Chapter VI.

Take the modulus of elasticity at 36,000,000 lbs. per square inch, then substituting in the formula

$$M = 2\pi N \frac{E I}{L}.$$

The moment required is $2 \times 3{\cdot}1416 \times 1\frac{1}{2} \times \frac{36{,}000{,}000 \times \frac{(\frac{1}{4})^3}{12}}{60 \times 12}$

$= 613{\cdot}7$ in. lbs., or $51{\cdot}1$ ft. lbs.

XVI.—The Resilience of a Spiral Spring of Rectangular Section.

Formula :— $$R = \tfrac{1}{6} f^2 \frac{V}{E};$$

where E is the modulus of elasticity of the material, f its proof stress; V the volume of metal in the spring and R the required resilience.

From the formula $N = \frac{p_1 L}{\pi E t}$ (see Chapter XIV.) we obtain

$$\pi N = \frac{f L}{E t};\ \text{(i.)};$$

N being now the number of turns required to produce in the spring an intensity of pressure amounting to f.

Also from the last chapter

$$M = 2\pi N \frac{E I}{L};\ \text{(ii.)}$$

Multiplying the equations (i.) and (ii.) together,

$$MN\pi = \frac{f L}{E t} 2\pi N \frac{E I}{L}$$

$$= 2\pi N \frac{f}{t} I$$

$$= 2 \frac{f L}{E t} \cdot \frac{f}{t} I \text{ from equation (i.)}$$

$$= 2 \frac{f^2}{t^2} \frac{L}{E} I.$$

But the spring being rectangular in section

$$I = \frac{h^3 b}{12};\ \text{(see Chapter VI.)}$$

in which h is equal to t, and b is the breadth of the section.

Therefore $\quad MN\,\pi = 2 \cdot \frac{f^2}{t^2} \cdot \frac{L}{E} \cdot \frac{t^3\,b}{12}$; (iii.)

$$= \tfrac{1}{6} \frac{f^2}{E} V,$$

since (L b t) is the volume of the spring.

But MN π is the energy absorbed, or work done, in winding the spring till strained to the elastic limit.

For if a given moment M is allowed to act through a given angle, the work done is, M multiplied by the angle.

In this case, a moment at first equal to nothing and finally equal to M, is made to act through the angle $2\,\pi$ N ; and taking the average value of M as $\frac{M}{2}$ and multiplying by $2\,\pi$ N, we obtain MN π.

Therefore, $\quad \tfrac{1}{6} f^2 \frac{V}{E}$ is the resilience of the spring.

COROLLARY :—When the spring is of circular section

$$I = \frac{\pi}{4} r^4 \text{ (see note at the end of Chapter V.)}$$

thus equation (iii.) of the above becomes

$$MN\,\pi = 2 \frac{f^2}{t^2} \cdot \frac{L}{E} \cdot \frac{\pi}{4} r^4$$

in which $t = 2r$.

Therefore, $\quad MN\,\pi = \tfrac{1}{8} f^2 \frac{L}{E} (\pi r^2)$

$$= \tfrac{1}{8} f^2 \frac{V}{E}$$

since πr^2 L is the volume of the spring.

EXAMPLE:—It is desired to connect to the axle of a tram car a spiral spring, to be wound up in stopping the tram ; the object being to restart the tram at almost its first velocity by releasing the spring.

If the weight of the tram without the spring is four tons when full, and its usual velocity four miles per hour; what weight of spring will be required?

We will suppose a steel spring of rectangular section, in which a working stress of 10 tons per square inch is permitted ; its modulus of elasticity 12,000 tons per square inch, and weight per cubic inch, 228 lbs. or ·00013 tons.

Four miles per hour is the velocity which a body would attain by falling 6·4 in. ; so that the spring must be able to absorb and store 6·4 W inch tons of work, where W is the weight of the tram, spring included.

Equating this to the resilience of spring,

$$6{\cdot}4\ \mathrm{W} = \tfrac{1}{6} f^2\ \frac{\mathrm{V}}{\mathrm{E}}$$

when f is the proof stress of the material.

But the stress permitted in the spring must not exceed the working stress, here taken at 10 tons per square inch.

Therefore,
$$6{\cdot}4\ \mathrm{W} = \tfrac{1}{6}\,(10^2)\ \frac{\mathrm{V}}{\mathrm{E}}$$

$$\tfrac{1}{6}\,(10)^2\ \frac{\dfrac{w}{\cdot 00013}}{12{,}000}$$

where w is the weight of the spring in tons.

Also
$$\mathrm{W} = (w + 4).$$

Therefore,
$$w + 4 = \frac{1}{6} \cdot \frac{100}{6{\cdot}4} \cdot \frac{w}{12{,}000 \times {\cdot}00013},$$

Whence,
$$w = \frac{-4}{1 - \dfrac{100}{6 \times 6{\cdot}4 \times 1{\cdot}2 \times 1{\cdot}3}} = \frac{-4}{1 - \dfrac{100}{60}}$$

$$= 6 \text{ tons nearly.}$$

Or the spring would weigh about half as much again as the tram.

XVII.—The Differential Equation to the Deformed Mean Fibre of a Beam.

Formula :—
$$d\,\frac{dy}{dx} = \frac{\mathrm{M}}{\mathrm{E\,I}}\ dx,$$

where I is the moment of inertia of the section of beam, and E the modulus of elasticity; also y and M are the deflection and bending moment respectively, at any place distance x from some point O in the beam.

Let O X represent the mean fibre of the beam in its original position when straight, and let the curve A B represent some portion of the same after loading has produced deformation (Fig. 26).

From the points A and B draw A C^1 and B C^1 tangents to the curve, meeting at the point C^1, also draw A C and B C at right angles to the latter, meeting at the point C, and produce A C^1 to B^1.

Now if A C^1 is inclined to O X at the angle θ, and C^1 B is inclined to

it at the angle θ^1, the angle B C^1 B^1, which is evidently the difference of these inclinations, is equal to $\theta^1 - \theta$.

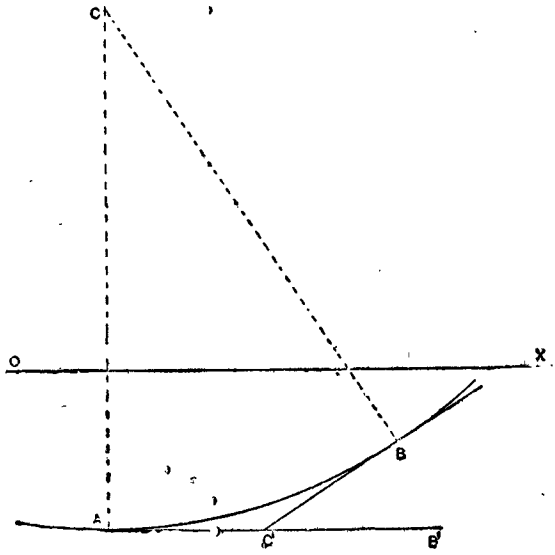

Fig. 26.

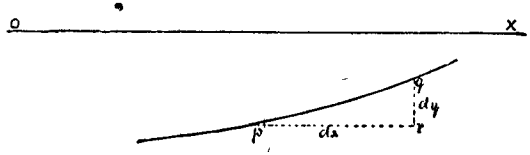

Fig. 27.

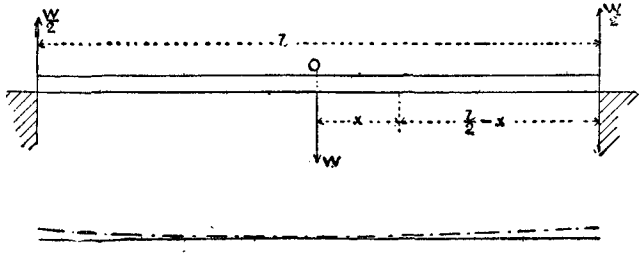

Fig. 28.

But the angle B C^1 B^1 is equal to the angle A C B, and if A B is not too great, so that it may be regarded as a small part of a circle, the angle A C B will be $\frac{A\,B}{R}$; where R is equal to A C or B C radii of the circle.

Therefore $$\theta^1 - \theta = \frac{A\,B}{R}$$

A B not being too large.

But if we regard the point A as distance x from O, and make A B *very small indeed*, as small as possible, so that it may be considered as dx or a small increase of x, then the angle B C^1 B may be regarded as $d\theta$ or a small increase of θ.

Thus the equation $$\theta^1 - \theta = \frac{A\,B}{R}$$

becomes $$d\,\theta = \frac{dx}{R},$$

and θ is the inclination of the deformed mean fibre to O X at a point in the beam distance x from O.

Let p be any such point, its distance from O is x, and its distance from the line O X is y (Fig. 27).

Draw an extremely short line pr parallel to O X, and rq at right angles to it.

Then pr and qr being both infinitesimal, may be called dx and dy respectively

Thus at the point p, the mean fibre is inclined to O X at the angle qpr, the tangent of which is $\frac{dy}{dx}$, but as the mean fibre of a beam is never bent very much, so this angle must always be extremely small; we may say that the angle itself is $\frac{dy}{dx}$.

Then substituting this in the above equation we obtain $d\,\frac{dy}{dx} = \frac{dx}{R}$.

So far geometrically.

But we have previously shown that R the radius of curvature of the mean fibre, at any point in the beam, is equal to $\frac{E\,I}{M}$. (See Chapter V.)

Hence $$d\,\frac{dy}{dx} = \frac{M}{E\,I}\,dx,$$ which is the required equation.

N.B.—This equation applies to the deformed mean fibre of a beam generally and under all circumstances; the ordinary equation to the deflected mean fibre in any particular case must be deduced from it; this is done in the next five chapters, by simply substituting for M the general expression for bending moment on the beam, and integrating.

The equation is sometimes written

$$\frac{d^2y}{dx^2} = \frac{M}{E\,I},$$

which means exactly the same thing, and may be at once translated into

$$d\,\frac{dy}{dx} = \frac{M}{E\,I}\,dx.$$

XVIII.—The Deflection of a Beam due to a Single Weight placed at the Centre.

Formula :—

$$y_o = \tfrac{1}{48}\,\frac{W\,l^3}{E\,I},$$

where E is the modulus of elasticity of the beam, l and I are its length of span and moment of inertia respectively ; also W is the amount of the load and y_o the deflection produced at the centre of span.

M, the bending moment at a point distance x from O the middle of the beam, is equal to

$$\frac{W}{2}\left(\frac{l}{2} - x\right), \text{ see Fig. 28.}$$

Substituting this value for M in the formula

$$d\,\frac{dy}{dx} = \frac{M}{E\,I}\,dx;$$

the equation to the deformed mean fibre, proved in Chapter XVII., we have

$$d\,\frac{dy}{dx} = \tfrac{1}{2} \cdot \frac{W}{E\,I}\left(\frac{l}{2} - x\right)dx.$$

Integrating both sides of the equation,

$$\int_0^{\frac{dy}{dx}} d\,\frac{dy}{dx} = \tfrac{1}{2} \cdot \frac{W}{E\,I}\int_0^{x}\left(\frac{l}{2}\,dx - x\,dx\right),$$

or

$$\frac{dy}{dx} = \tfrac{1}{2}\,\frac{W}{E\,I}\left(\frac{l}{2}\,x - \frac{x^2}{2}\right),$$

multiplying by dx,

$$dy = \tfrac{1}{2}\,\frac{W}{E\,I}\left(\frac{l}{2}\,x - \frac{x^2}{2}\right)dx.$$

Integrating both sides again,

$$\int_0^y dy = \tfrac{1}{2}\frac{W}{EI}\int_0^x \left(\frac{l}{2}\,x\,dx - \frac{x^2}{2}\,dx\right),$$

or

$$y = \tfrac{1}{2}\frac{W}{EI}\left(\frac{l}{2}\cdot\frac{x^2}{2} - \tfrac{1}{2}\cdot\frac{x^3}{3}\right).$$

This is the ordinary equation to the deformed mean fibre, for the particular case in which the beam is loaded with a single weight at the middle of the span. By giving x the values 1 ft., 2 ft., 3 ft., etc., successively, the corresponding values of y may be found, and the curve of the deflected mean fibre of the beam plotted out, as sketched below the figure.

When x is made equal to $\frac{l}{2}$, y evidently becomes the difference of level between the highest and lowest parts of the curve, which is the *deflection* at the centre of span.

Hence we obtain $y_o = \tfrac{1}{2}\frac{W}{EI}\left(\tfrac{1}{2}\cdot\frac{1}{4\times 2} - \tfrac{1}{2}\cdot\frac{1}{3\times 8}\right)l^3,$

$$= \tfrac{1}{48}\frac{W\,l^3}{EI}.$$

This takes no account of any deflection due to the weight of the beam itself, which must be calculated, when needed, by the formula in Chapter XXI., and added on to that obtained by the above.

COROLLARY :—(1.) The maximum bending moment on the beam occurs at the middle cross section, and is equal to $\frac{Wl}{4}$; this must not exceed in amount $\frac{p_1}{X}$ I (see Chapter V.), where p_1 is the working intensity of pressure for the material of which the beam is composed.

From the equation $$\frac{Wl}{4} = \frac{p_1}{X}\,I,$$

we have $$W = \frac{4\,p_1\,I}{Xl};$$

the greatest safe load for the beam.

Substituting this value of W in the formula

$$y_o = \tfrac{1}{48}\frac{Wl^3}{EI},$$

we obtain

$$y_1 = \tfrac{1}{12}\frac{p_1\,l^2}{X\,E}$$

y_1 being thus written for the greatest safe value of y_o.

COROLLARY :—(2.) If we load the beam gradually, putting on the central weight bit by bit, so that it sinks down to y_o, the work done will be expressed by

$$\int_o^W y_o \, d\,W,$$

being the sum of all the small amounts of work done at each addition of load.

But $$y = \tfrac{1}{48} \frac{W\,l^3}{E\,I}.$$

Therefore the work done

$$= \tfrac{1}{48} \cdot \frac{l^3}{E\,I} \int_o^W W \, d\,W,$$

$$= \tfrac{1}{96} \frac{l^3}{E\,I} W^2, \quad (1.)$$

$$= \frac{y_o}{2} W.$$

When the load W is made sufficient to raise an intensity of pressure on the section of maximum bending moment, amounting to f, the proof stress of the material, we have for a cross section at the centre of span (where the maximum bending moment occurs)

$$\frac{W\,l}{4} = \frac{f}{X} I,$$

being an equation of bending moment to moment of resistance.

Whence $$W = \frac{4\,f\,I}{X\,l}.$$

Substituting this value of W in the equation marked (1) above,

the work done $$= \tfrac{1}{96} \cdot \frac{l^3}{E\,I} \left(\frac{4\,f\,I}{X\,l} \right)^2$$

$$= \tfrac{1}{6} \cdot \frac{f^2\,l}{E\,X} \cdot I;$$

which (since the proof strain has been reached) is the resilience of the beam.

NOTE :—In this chapter the moment of inertia is assumed to be constant throughout the length of the beam.

EXAMPLE :—An opening of 30 ft. is spanned by a rolled iron beam, the moment of inertia of which is 2,514·5 in inches. It is proposed to load it at the centre with a weight of 13 tons ; what deflection may be expected, the modulus of elasticity being taken at 11,100 tons per square inch ?

By formula $$y_o = \tfrac{1}{48} \frac{W\,l^3}{E\,I}.$$

The deflection at the centre is $\frac{1}{48} \cdot \frac{13\ (30 \times 12)^3}{11{,}100 \times 2{,}514{\cdot}5}$

$= {\cdot}45$ inches.

When beams have to be laid to support a flooring oblong in plan, they are usually placed as in Fig. 29.

If, however, the floor will be chiefly called upon to sustain shocks, such as might be caused by the falling of a heavy body, it is better to lay the beams as in Fig. 30, *even though the cross section of the beams and their distance apart are the same as before.*

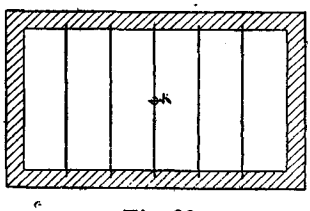

Fig. 29.

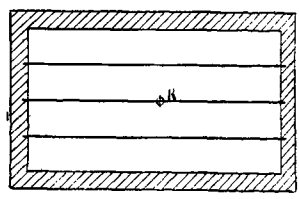

Fig. 30.

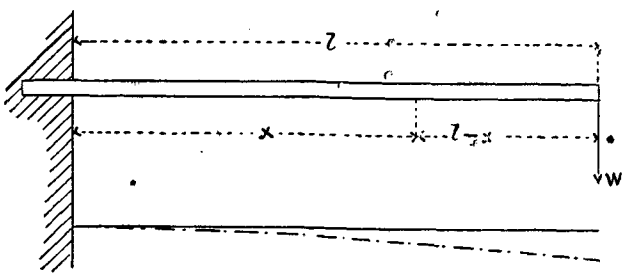

Fig. 31.

For let a and b be the lengths of the sides of the floor, then a shock administered at the point k in the middle would in the first instance be sustained by a beam of length a, and in the second by one of length b; the resiliences of which are as

$$\tfrac{1}{6} \frac{f^2\, a}{\mathrm{E\,X}}\ \mathrm{I} \text{ is to } \tfrac{1}{6} \frac{f^2\, b}{\mathrm{E\,X}}\ \mathrm{I}.$$

Thus the latter arrangement (though far weaker in all other respects) increases the power of the floor to withstand such a shock, in the proportion of b to a. And the weight of material used for the beams is in both cases the same.

XIX.—The Deflection of a Cantilever due to a Single Weight placed at the Free End.

Formula :— $$y_o = \tfrac{1}{3}\,\frac{W\,l^3}{E\,I},$$

where E is the modulus of elasticity of the beam, l and I are its length and the moment of inertia of its section, respectively; also W is the amount of the load, and y the deflection produced at the free end.

M, the bending moment at a point distance x from the fixed end of the beam, is equal to

$$W\,(l - x);$$ see Fig. 31.

Substituting this value for M in the formula

$$d\,\frac{dy}{dx} = \frac{M}{E\,I}\,dx,$$

the equation to the deformed mean fibre proved in Chapter XVII., we have

$$d\,\frac{dy}{dx} = \frac{W}{E\,I}\,(l - x)\,dx.$$

Integrating both sides of the equation,

$$\int_0^{\frac{dy}{dx}} d\,\frac{dy}{dx} = \frac{W}{E\,I}\int_0^{x}\left(l\,dx - x\,dx\right),$$

or $$\frac{dy}{dx} = \frac{W}{E\,I}\left(l\,x - \frac{x^2}{2}\right),$$

multiplying by dx, $$dy = \frac{W}{E\,I}\left(l\,x - \frac{x^2}{2}\right)dx.$$

Integrating both sides again,

$$\int_0^{y} dy = \frac{W}{E\,I}\int_0^{x}\left(l\,x\,dx - \tfrac{1}{2}\,x^2\,dx\right),$$

or $$y = \frac{W}{E\,I}\left(l\,\frac{x^2}{2} - \tfrac{1}{2}\cdot\frac{x^3}{3}\right).$$

This is the ordinary equation to the deformed mean fibre, for the particular case in which the beam is a cantilever deflected by a single load placed at the extremity. By giving x the values 1 ft., 2 ft., 3 ft., etc., successively, the corresponding values of y may be found, and the curve of the deflected mean fibre of the beam plotted out, as sketched below the figure.

When x is made equal to l, y evidently becomes the difference of

63762

level between the highest and lowest parts of the curve, which is the *deflection* at the free end.

We thus obtain $$y_o = \frac{W}{E I}\left(\tfrac{1}{2} - \tfrac{1}{6}\right) l^3,$$

$$= \tfrac{1}{3} \cdot \frac{W l^3}{E I}.$$

This again takes no account of any deflection due to the weight of the beam itself, which must be calculated, when needed, by the formula given in Chapter XX., and added on to that obtained by the above.

COROLLARY :—(1.) The maximum bending moment on the beam occurs at the fixed end, and is equal to $W l$; this must not exceed in amount $\frac{p_t}{X} I$ the moment of resistance (see Chapter V., Corollary), where p_t is the working intensity of stress for the material of which the beam is composed.

From the equation $$W l = \frac{p_t}{X} I$$

we have $$W = \frac{p_t I}{X l},$$

the greatest safe load for the beam.

Substituting this value of W in the formula

$$y_o = \tfrac{1}{3} \frac{W l^3}{E I},$$

we obtain $$y_t = \tfrac{1}{3} \frac{p_t l^2}{X E},$$

y_t being here used to denote the greatest safe value for y_o.

COROLLARY :—(2.) If we load the beam gradually, putting on the weight at the end bit by bit, so that it sinks down to y_o, the work done will be expressed by $\int_o^W y_o \, d\,W$, being the sum of all the small amounts of work done by each additament to the load.

But $$y_o = \tfrac{1}{3} \frac{W l^3}{E I}.$$

Therefore the work done $$= \tfrac{1}{3} \cdot \frac{l^3}{E I} \int_o^W W \, d\,W$$

$$= \tfrac{1}{6} \cdot \frac{l^3}{E I} W^2, \quad (1)$$

$$= \frac{y_o}{2} W.$$

When the load W is made sufficient to raise the intensity of pressure on the section of maximum bending moment to f, the proof strain of

the material, we have for a cross section, at the fixed end (where the maximum bending moment occurs),

$$W\,l = \frac{f}{X}\,I \text{ (see Chapter V.)},$$

being an equation of bending moment to moment of resistance.

Whence $$W = \frac{f\,I}{X\,l}.$$

Substituting this value for W in the equation marked (1) above,

the work done $$= \tfrac{1}{6}\frac{l^3}{E\,I}\left(\frac{f\,I}{X\,l}\right)^2$$

$$= \tfrac{1}{6}\cdot\frac{f^2\,l}{E\,X}\,I,$$

which (since the proof stress has been reached) is the resilience of the beam.

NOTE :—In this Chapter also, I the moment of inertia is assumed to be the same throughout the length of the beam.

EXAMPLE :—A wooden cantilever, 5 in. square in section, projects from the wall of a building, and carries a signboard at its extremity ; it is stayed vertically, but not horizontally, and in high winds a lateral deflection of $1\frac{1}{2}$ in. may be observed.

Take E at 700 tons per square inch, and discuss safety.

From the formula, $$y_1 = \frac{p_1\,l^2}{3\,X\,E},$$

we have $$p_1 = \frac{3\,y_1\,E\,X}{l^2}$$

$$= \frac{3 \times 1\frac{1}{2} \times 700 \times 2\frac{1}{2}}{(6 \times 12)^2}$$

$$= 1\tfrac{1}{2} \text{ tons per square inch.}$$

This is too high for a working stress, so if lateral stays are inconvenient, it might be proposed to attach the signboard nearer in to the wall.

From Corollary (2), however, it will be seen that such a course would reduce the resilience of the beam, and so lessen its power to withstand the shock of a sudden gust.

The arrangement would not *necessarily* be stronger.

XX.—The Deflection of a Cantilever due to a Load Uniformly Distributed.

Formula :— $$y_o = \tfrac{1}{8} \frac{W\, l^3}{E\, I},$$

where E is the modulus of elasticity of the beam, l and I are its length and the moment of inertia of its section respectively; also W is the whole weight of the distributed load, and y_o the deflection produced at the free end.

Let w denote the amount of W per foot run.

Then M, the bending moment at a point distance x from the fixed end of the beam, is equal to

$$w\,(l - x)\,\frac{l - x}{2} \text{ (see Fig. 32),}$$

or $$\frac{w}{2}\,(l - x)^2.$$

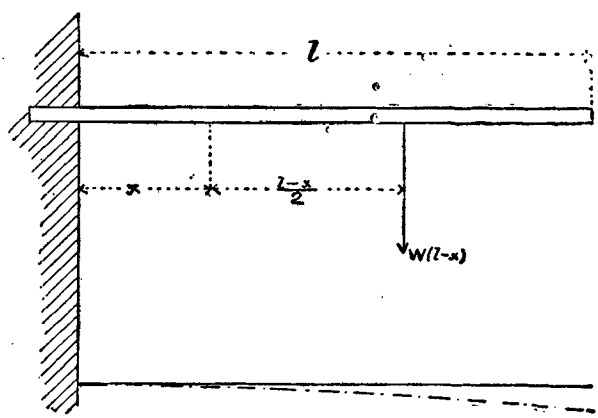

Fig. 32.

Substituting this value for M in the formula,

$$d\,\frac{dy}{dx} = \frac{M}{E\,I}\,dx,$$

the equation to the deformed mean fibre, proved in Chapter XVIII., we have

$$d\,\frac{dy}{dx} = \tfrac{1}{2} \cdot \frac{w}{E\,I}\,(l - x)^2\,dx,$$

both sides of the equation.

$$\int_0^{\frac{dy}{dx}} d\,\frac{dy}{dx} = \tfrac{1}{2}\,\frac{w}{\mathrm{E\,I}} \int_0^{x} (l^2\,dx - 2\,l\,x\,dx + x^2\,dx),$$

or $$\frac{dy}{dx} = \tfrac{1}{2}\,\frac{w}{\mathrm{E\,I}}\left(l^2\,x - 2\,l\,\frac{x^2}{2} + \frac{x^3}{3}\right),$$

multiplying by dx, $$dy = \tfrac{1}{2}\,\frac{w}{\mathrm{E\,I}}\left(l^2\,x - l\,x^2 + \frac{x^3}{3}\right)dx,$$

integrating both sides again,

$$\int_0^{y} dy = \tfrac{1}{2}\,\frac{w}{\mathrm{E\,I}} \int_0^{x} (l^2\,x\,dx - l\,x^2\,dx + \tfrac{1}{3}\,x^3\,dx),$$

or $$y = \tfrac{1}{2}\,\frac{w}{\mathrm{E\,I}}\left(l^2\,\frac{x^2}{2} - l\,\frac{x^3}{3} + \tfrac{1}{3}\cdot\frac{x^4}{4}\right).$$

This is the ordinary equation to the deformed mean fibre, for the particular case in which the beam is a cantilever, and the load distributed uniformly. By giving to x the values 1 ft., 2 ft., 3 ft., etc., successively, the corresponding values of y may be found, and the curve of the deflected mean fibre of the beam plotted out, as sketched in Fig. 32.

When x is made equal to l, y evidently becomes the difference of level between the highest and lowest parts of the curve, which is the deflection of the free end.

We thus obtain $$y_o = \tfrac{1}{2}\cdot\frac{w}{\mathrm{E\,I}}\,(\tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{12})\,l^4$$

$$= \tfrac{1}{8}\cdot\frac{\mathrm{W}\,l^3}{\mathrm{E\,I}}; \quad \because w\,l = \mathrm{W}.$$

Here the weight of the beam itself may be included in the estimation of w, if necessary.

COROLLARY :—The maximum bending moment on the beam occurs at the fixed end, and is equal to $\frac{\mathrm{W}\,l}{2}$; this must not exceed in amount $\frac{p_1}{\mathrm{X}}\mathrm{I}$ (see Chapter V.), where p_1 is the working intensity of stress for the material of which the beam is composed.

From the equation, $$\frac{\mathrm{W}\,l}{2} = \frac{p_1}{\mathrm{X}}\,\mathrm{I},$$

we have $$\mathrm{W} = \frac{l\,p_1\,\mathrm{I}}{\mathrm{X}\,l},$$

the greatest safe load for the beam. Substituting this value of W in the formula,

$$y_o = \tfrac{1}{8}\,\frac{\mathrm{W}\,l^3}{\mathrm{E\,I}},$$

we obtain $$y_1 = \tfrac{1}{4} \cdot \frac{p_1 \, l^2}{X \, E},$$

y_1 being used to denote the highest safe value of y_0.

EXAMPLE :—A piece of Tee iron, 4 in. $\times$ 4 in. $\times$ $\frac{1}{2}$ in., projects 7 ft. as a cantilever; the total load upon it amounts to 8 cwt., distributed uniformly.

Calculate the deflection.

It is necessary first to estimate the value of I the moment of inertia. To do this we must begin by finding the position of the neutral axis of the section. Let X_1 be its distance from the top (see Fig. 33).

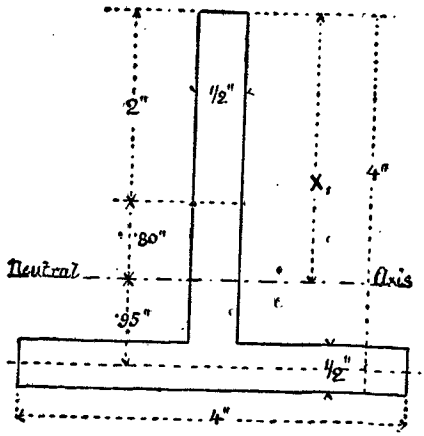

Fig. 33.

Taking moments of area,

$$X_1 (4 \times \tfrac{1}{2} + 3\tfrac{1}{2} \times \tfrac{1}{2}) = 2 (4 \times \tfrac{1}{2}) + 3\tfrac{3}{4} (3\tfrac{1}{2} \times \tfrac{1}{2}),$$

or $$2\tfrac{7}{4} X_1 = 4 + \tfrac{105}{16},$$

$$X_1 = \frac{169 \times 4}{16 \times 15} = 2{\cdot}8 \text{ inches.}$$

Therefore, the neutral axis is 2·8 in. from the top of the section.

The moment of inertia is now found by applying the formula $I_m = I_n + D^2 S$, to both flanges consecutively (see Chapter VIII.).

$$I = \frac{4^3 \times \frac{1}{2}}{12} + (4 \times \tfrac{1}{2}) \, (\cdot 8)^2 + \frac{(\frac{1}{2})^3 \times 3\frac{1}{2}}{12} + (\tfrac{1}{2} \times 3\tfrac{1}{2}) \, (\cdot 7)^2$$

$$= 2{\cdot}666 + \cdot 96 + \cdot 036 + 1{\cdot}579$$

$$= 5{\cdot}96 \text{ in inches.}$$

Substituting now in the formula,

$$y_o = \tfrac{1}{8} \cdot \frac{W\,l^3}{E\,I},$$

taking E at 13,000 tons per square inch.

The deflection at the extremity of the Tee iron will amount to

$$\frac{(8/20)\ (7 \times 12)^3}{8 \times 13{,}000 \times 5{\cdot}96} = {\cdot}385 \text{ inch.}$$

XXI.—The Deflection of a Beam, due to a Load, Uniformly Distributed.

Formula :— $$y_o = \frac{5}{384} \cdot \frac{W\,l^3}{E\,I},$$

where E is the modulus of elasticity of the beam, l and I are its length of span and the moment of inertia of its section respectively, W the total weight of the distributed load, and y_o the deflection produced at the centre of span.

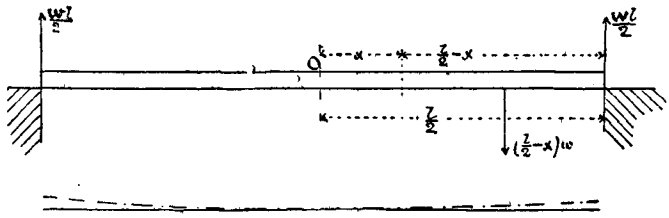

Fig. 34.

Let w denote the amount of W per foot run, then M, the bending moment, at a point distance x from O, the middle of the beam, is equal to,

$$\frac{w\,l}{2}\left(\frac{l}{2} - x\right) - \left(\frac{l}{2} - x\right) w\,\tfrac{1}{2}\left(\frac{l}{2} - x\right), \text{ see Fig. 34,}$$

$$= \frac{w}{2}\left(\frac{l}{2} - x\right)\left(l - \frac{l}{2} + x\right),$$

$$= \frac{w}{2}\left(\frac{l}{2} - x\right)\left(\frac{l}{2} + x\right) = \frac{w}{2}\left(\frac{l^2}{4} - x^2\right).$$

Hence, substituting for M in formula

$$d\,\frac{dy}{dx} = \frac{M}{E\,I}\,dx,$$

the equation to deformed mean fibre proved in Chapter XVII.

$$d\left(\frac{dy}{dx}\right) = \tfrac{1}{2}\,\frac{w}{E\,I}\left(\frac{l^2}{4} - x^2\right)dx.$$

Integrating both sides of the equation

$$\int_0^{\frac{dy}{dx}} d\left(\frac{dy}{dx}\right) = \tfrac{1}{2}\frac{w}{E\,I}\cdot\frac{l^2}{4}\int_0^x dx - \tfrac{1}{2}\frac{w}{E\,I}\int_0^x x^2\,dx.$$

Therefore $$\frac{dy}{dx} = \tfrac{1}{2}\frac{w}{E\,I}\cdot\frac{l^2}{4}x - \tfrac{1}{2}\frac{w}{E\,I}\cdot\frac{x^3}{3},$$

then dividing through by dx

$$dy = \tfrac{1}{2}\cdot\frac{w}{E\,I}\cdot\frac{l^2}{4}\cdot x\,dx - \tfrac{1}{2}\cdot\frac{w}{E\,I}\cdot\frac{x^3}{3}\cdot d\,x,$$

integrating both sides again

$$\int_0^y dy = \cdot\tfrac{1}{2}\cdot\frac{w}{E\,I}\cdot\frac{l^2}{4}\int_0^x x\,dx - \tfrac{1}{2}\frac{w}{E\,I}\int_0^x \frac{x^3}{3}\,dx,$$

or $$y = \tfrac{1}{2}\frac{w}{E\,I}\cdot\frac{l^2}{4}\cdot\frac{x^2}{2} - \tfrac{1}{2}\frac{w}{E\,I}\cdot\frac{x^4}{12},$$

$$= \tfrac{1}{2}\frac{w}{E\,I}\left(\frac{l^2}{4}\cdot\frac{x^2}{2} - \frac{x^4}{12}\right).$$

This is the ordinary equation to the deformed mean fibre in the particular case now considered, where the beam is deflected by a load distributed uniformly. By giving x the values 1 ft., 2 ft., 3 ft., etc., successively, the corresponding values of y may be found, and the curve of the deflected mean fibre of the beam plotted out, as sketched below the figure.

When x is made equal to $\frac{l}{2}$, y evidently becomes the difference of level between the highest and lowest parts of the curve, which is the *deflection* at the centre of span.

We obtain thus :—

$$y_0 = \tfrac{1}{2}\frac{w}{E\,I}\left(\tfrac{1}{4}\cdot\tfrac{1}{8} - \tfrac{1}{192}\cdot\tfrac{1}{16}\right)l^4,$$

$$= \tfrac{5}{384}\frac{W\,l^3}{E\,I};\ \therefore w\,l = W.$$

Here, the weight of the beam may be included in the estimation of w, if desired.

COROLLARY :—(1.) The maximum bending moment on the beam occurs at the middle cross section, and is equal to $\frac{W\,l}{8}$; this must not exceed in amount $\frac{p_1}{x}$ I ; (see Chapter V.), where p_1 is the working intensity of stress for the material of which the beam is composed.

For the equation $\frac{W\,l}{8} = \frac{p_1}{X}\,I$,

we have $W = \frac{8\,p_1\,I}{X\,l}$; the greatest safe load for the beam.

Substituting this value of W is the formula

$$y_o = \tfrac{5}{384}\,\frac{W\,l^3}{E\,I},$$

we obtain $y_1 = \tfrac{5}{48}\,\frac{p_1\,l^2}{X\,E}$; y_1 being here used to denote the greatest safe value of y_o.

COROLLARY :—(2.) The value of E may be readily determined in a rough manner, from an observation of the deflection produced by loading a beam of the material in question.

Thus, from formula $y_o = \tfrac{5}{384}\,\frac{W\,l^3}{E\,I}$,

we have, $E = \tfrac{5}{384}\,\frac{W\,l^3}{y_o\,I}$

$= \tfrac{5}{32}\,\frac{W\,l^3}{y_o\,h^3\,b}$, when the beam is of rectangular section, see Chapter VI.

EXAMPLE :—The drawings of a proposed structure show a beam composed of two Memel deals, scantling 9 in. × 3 in., with a ½ inch wrought iron between them. Figure 35 is a cross section of the beam, whose span is 17 ft. Discuss the merits of the design.

Take E at 700 tons per square inch for the wood, and 12,500 tons per square inch for the iron.

Also take p_1, the working intensity of stress, at ¾ tons per square inch for the wood, and 5 tons per square inch for the iron.

From the formula $y_1 = \tfrac{5}{48}\,\frac{p_1\,l^2}{X\,E}$,

the greatest amount to which the iron plate can be permitted to deflect is

$$\tfrac{5}{48} \cdot \frac{5\,(17 \times 12)^2}{4\tfrac{1}{2} \times 12{,}500} = \cdot 4 \text{ inches.}$$

Therefore the whole composite beam must not be loaded more than is sufficient to produce a deflection of ·4 in. at the centre of span.

From the formula

$$y_o = \tfrac{5}{384}\,\frac{W\,l^3}{E\,I},$$

we obtain $W = \tfrac{384}{5}\,y_o\,\frac{E\,I}{l^3}$.

This gives us the load taken up by a beam when the deflection at centre of span amounts to y_o. Applying it to the central iron plate when deflected ·4 inches

we obtain, $$W = \tfrac{384}{5} \times \cdot 4 \times \frac{12{,}500 \times \frac{9^3 \times \frac{1}{2}}{12}}{(17 \times 12)^3}$$

$$= 1{\cdot}37 \text{ tons.}$$

Applying the same formula to the two wooden parts of the beam which are of necessity deflected to the same amount,

$$W = \tfrac{384}{5} \times \cdot 4 \times \frac{700 \times \frac{9^3 \times 6}{12}}{(17 \times 12)^3}$$

$$= \cdot 92 \text{ tons.}$$

The complete beam therefore should not bear more than the sum of these two values for W, *i.e.*, 2·29 tons evenly distributed.

Let us now find what the wood would bear alone, supposing the beam put up without the iron plate.

The bending moment at the centre of span, *i.e.*, the greatest bending moment produced by a distributed load W is $\frac{W\,l}{8}$.

Therefore, from the formula

$$M = \frac{p_1}{X} I \qquad \text{(see Chapter V.)},$$

$$\frac{W\,l}{8} = \frac{p_1}{X} I,$$

or $$W = \frac{8}{l} \frac{p_1}{X} I.$$

Substituting, $$W = \frac{8 \times \frac{3}{4} \times \frac{9^3 \times 6}{12}}{(17 \times 12 \times 4\frac{1}{2})}$$

$$= 2{\cdot}38 \text{ tons.}$$

This value of W is greater than the sum of the other two; so the introduction of the iron plate does not *strengthen* the beam at all; its effect is only to *stiffen* it, *i.e.*, reduce the deflection under a given load.

For the deflection of the wood alone under a load of 2·38 tons by the formula

$$y_0 = \tfrac{5}{384} \cdot \frac{W\,l^3}{E\,I}$$

would be $$\tfrac{5}{384} \cdot \frac{2{\cdot}38 \times (17 \times 12)^3}{700 \times \frac{9^3 \times 6}{12}} = 1{\cdot}03 \text{ in.}$$

When it is desired to combine two materials in this way for purposes of *strength* and *economy*, the design must be such that both are subjected to their respective working stresses simultaneously.

Figure 36 indicates an arrangement which fulfils this condition. By equating the general value of y_1 for the wood, to the value of y_1 for the iron, we may obtain the ratio of the values of X; which is the same as the ratio of the depths of the two materials.

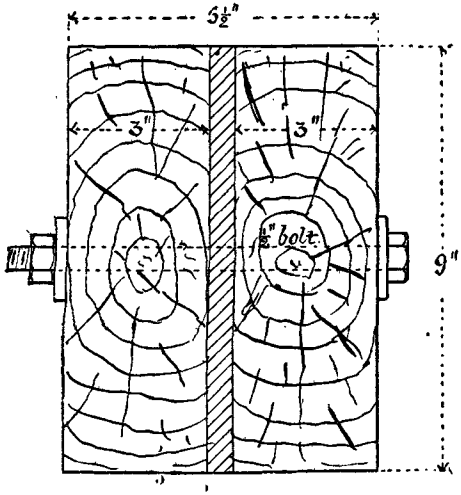

Fig. 35.

Thus introducing the suffixes w and r to distinguish the quantities which refer to the wood, from those which refer to the iron, we have

$$\tfrac{5}{48} \cdot \frac{p_{1w}\, l^2}{X_w\, E_w} = \tfrac{5}{48} \cdot \frac{p_{1r}\, l^2}{X_r\, E_r},$$

whence $$\frac{X_r}{X_w} = \frac{p_{1r}\, E_w}{p_{1w}\, E_r},$$ and this should be the ratio of the depths of the iron and wood in all cases.

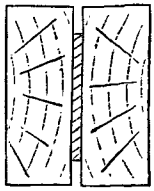

Fig. 36.

XXII.—The Safe Working Deflection for Built Girders of Uniform Depth.

Formula :— $$y_1 = \frac{p_1 l^2}{4 E D}$$

where l is the length of the girder, E the modulus of elasticity of the materials, p_1 the working stress, when the same is used in both tension and compression, or the mean working stress when it differs for the two booms.

Also D is the depth of the girder, which must be the same throughout, and y_1 the safe deflection at centre of span.

M, the bending moment on any cross section of the girder, is equal to $\frac{p_0}{X}$ I, the moment of resistance of that cross section. (See Chapter V., Corollary.)

Moreover, a built girder will be designed with this object:—that under a full load, p_0 of the above expression shall, in all sections, as far as possible, amount to p_1, the working stress.

Hence, $$M = \frac{p_1}{X} I \text{ under the full load.}$$

Substituting this value of M in the formula

$$d \frac{dy}{dx} = \frac{M}{EI} dx,$$

the equation to the deformed mean fibre, proved in Chapter XVII., we have

$$d \frac{dy}{dx} = \frac{p_1}{EX} dx.$$

Here X is constant, since the depth of the girder is the same throughout. Hence integrating both sides,

$$\int_0^{\frac{dy}{dx}} d \frac{dy}{dx} = \frac{p_1}{EX} \int_0^x dx$$

or $$\frac{dy}{dx} = \frac{p_1 x}{EX};$$

multiplying by dx $$dy = \frac{p_1}{EX} x \, dx,$$

integrating again

$$\int_0^y dy = \frac{p_1}{EX} \int_0^x x \, dx,$$

or $$y = \frac{p_1}{EX} \cdot \frac{x^2}{2};$$

the equation to the mean fibre of the girder, when deflected by the full load it was designed to carry.

The curve may be plotted out in the usual way, by giving x the values 1 ft., 2 ft., 3 ft., etc., successively, and determining the corresponding values of y.

When x is made equal to $\frac{l}{2}$, y becomes the difference of level between the middle and ends of the curve, and is therefore equal to the deflection at the centre.

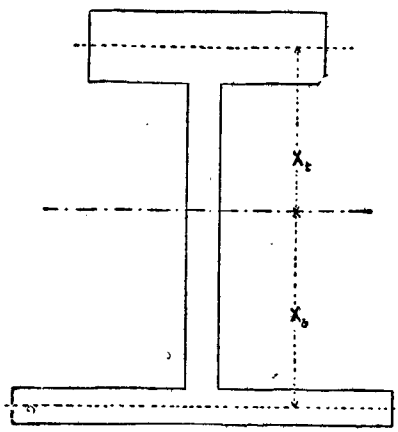

Fig. 37.

Whence $$y_1 = \frac{p_1 l^2}{E X 8};$$

for this being the proper deflection, under the full load for which the girder was designed, is the greatest deflection for which it is safe.

When the two booms of the girder are equal and similar $X = \frac{1}{2} D$, and the formula becomes at once

$$y_1 = \frac{p_1 l^2}{4 E D};$$

but when the two beams are unequal, that is, when the working stresses used in tension and compression are different, it must be remembered that X is the distance of the neutral axis from that extremity of the cross section at which the stress p_1 obtains. See Chapter IV.

It does not matter which end of the cross section is considered, for the stress on any part being proportional to its distance from the neutral axis

$$\frac{p_t}{X_t} = \frac{p_b}{X_b};$$

the suffixes t and b indicating top and bottom boom respectively, see Fig. 36.

Hence $$y_1 = \frac{p_t\, l^2}{\mathrm{E}\, \mathrm{X}_t\, 8} = \frac{p_b\, l^2}{\mathrm{E}, \mathrm{X}_b\, 8};$$

which is equal to

$$\frac{\frac{1}{2}\,(p_t + p_b)\, l^2}{4\,\mathrm{E}\,\mathrm{D}} \text{ or } \frac{p_1\, l^2}{4\,\mathrm{E}\,\mathrm{D}},$$

where p_1 is the mean of pt and p_b as described at the beginning.

The last may be shown as follows:—

$$\frac{p_t}{\mathrm{X}_t} = \frac{p_b}{\mathrm{X}_b}$$

$$\therefore \frac{p_b}{p_t} = \frac{\mathrm{X}_b}{\mathrm{X}_t}$$

Adding unity to both sides, and dividing by 2

$$\frac{p_b + p_t}{2\,p_t} = \frac{\mathrm{X}_t + \mathrm{X}_b}{2\,\mathrm{X}_t}$$

$$\frac{\frac{1}{2}\,(p_b + p_t)}{\mathrm{X}_t + \mathrm{X}_b} = \frac{p_t}{2\,\mathrm{X}_t}.$$

Thus $\frac{p_t}{2\,\mathrm{X}_t}$ is equal to the mean stress divided by $\mathrm{X}_t + \mathrm{X}_b$ the depth of the girder.

NOTE :—It is obvious that this formula is only approximate, for many reasons. Also, the calculated value of y_1 must *necessarily* include the deflection due to the weight of the girder itself, which is not observed when the girder is in position.

COROLLARY :—From the formula

$$y_1 = \frac{p_1\, l^2}{4\,\mathrm{E}\,\mathrm{D}};$$

$$\mathrm{D} = \frac{p_1\, l^2}{4\,\mathrm{E}\, y_1}.$$

This determines the minimum depth of a girder, when it is desired that the working deflection shall not exceed y.

Further, $$\frac{\mathrm{D}}{l} = \frac{p_1}{4\,\mathrm{E}} \cdot \frac{l}{y_1}$$

taking the working deflection at $\frac{1}{1600}$ of the span, a common practice,

$$\frac{l}{y_1} = 1{,}600.$$

Then $$\frac{\mathrm{D}}{l} = 400\ \frac{p_1}{\mathrm{E}},$$

which gives the correct proportion of the depth to the length of a girder.

EXAMPLE :—On inspecting an iron girder of uniform depth, 8 ft.,

spaning 68 ft., a deflection of one inch is observed. The girder was designed for a working of 4 tons per square inch in the top boom, and 5 tons per square inch in the bottom. Take E at 12,500 tons per square inch and discuss safety.

From the formula,

$$y_1 = \frac{p_1 l^2}{4 \text{ E D}}, \text{ or } \frac{\frac{1}{2}(p_t + p_b) l^2}{4 \text{ E D}}.$$

The safe working deflection should be

$$\frac{4\frac{1}{2}(68 \times 12)^2}{4 \times 12{,}500\ (8 \times 12)},$$

$$= \cdot 67 \text{ inches.}$$

Thus we may conclude that the girder is over-strained by about 50 per cent.

XXIII.—The Deflection of Carriage Springs.

Formula :— $$y_o = \frac{3 \text{ W } l^3}{n \text{ E } t^3 b},$$

where l is the length of each branch of the spring from buckle to point, n is the number of plates in the spring, and t the thickness of each plate ; also E is the modulus of elasticity, and b the breadth of the the spring.

When the number of plates of which the spring is composed is considerable, we may suppose the change of form in cross-section, which takes place from buckle to finish, to proceed continuously ; the method by integration can then be used without material inaccuracy, and the above formula obtained from the equation

$$d\frac{dy}{dx} = \frac{\text{M}}{\text{E I}} dx$$

(proved in Chapter XVII.) in much the same manner as before.

At a point distance x from the buckle, the bending moment is $\frac{1}{2}$ W $(l - x)$, see Fig. 37.

So we have $$d\frac{dy}{dx} = \frac{\frac{1}{2} \text{ W } (l - x)}{\text{E I}} dx.$$

But in this case I is not constant, hence on integrating it cannot be placed outside the sign of integration as heretofore.

Since the offset of each plate in the spring is the same, and equal to

D (see Fig. 38), the number of plates which will appear in a section at distance x from the buckle, will be $\frac{l-x}{D}$, approximately. Also since the several plates can slide over each other, the moment of inertia of the whole section can only be made up of the moments of inertia separately, of each plate contained in it.

Hence $$I = \frac{l-x}{D} \cdot \frac{t^3 b}{12};$$

for the plates are supposed to be equal in sectional area and $\frac{t^3 b}{12}$ is the moment of inertia of each of them.

Therefore $$d\frac{dy}{dx} = \frac{\frac{1}{2} W (l-x)}{E \frac{t^3 b (l-x)}{12 D}} dx = \frac{6 W D}{E t^3 b} dx$$

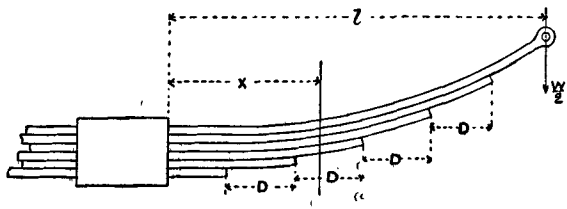

Fig. 38

Integrating both sides of the equation,

$$\int_0^{\frac{dy}{dx}} d\frac{dy}{dx} = \frac{6 W D}{E t^3 b} \int_0^x dx$$

or $$\frac{dy}{dx} = \frac{6 W D}{E t^3 b} x.$$

Again $$\int_0^y dy = \frac{6 W D}{E t^3 b} \int_0^x x\, dx.$$

Hence $$y = \frac{6 W D}{E t^3 b} \cdot \frac{x^2}{2}.$$

$$= \frac{3 W l x^2}{E t^3 b n} \text{ because } D = \frac{l}{n}.$$

And when x is made equal to l; y becomes y_o, the deflection at the free end of the spring.

Therefore $$y_o = \frac{3\ W\ l^3}{E\ i^3\ b\ n}.$$

NOTE :—This formula can only be approximate for many reasons, notably on account of friction, which must take place between adjacent plates of the spring.

XXIV.—THE THEOREM OF THREE MOMENTS.

Formula :—

$$4\ M_A l_1 + 8\ M_B (l_1 + l_2) + 4\ M_C l_2 = -w_1 l_1 - w_2 l_2^3,$$

where M_A M_B M_C are bending moments on a continuous girder at any three consecutive points of support; also l_1 l_2 are the lengths of the

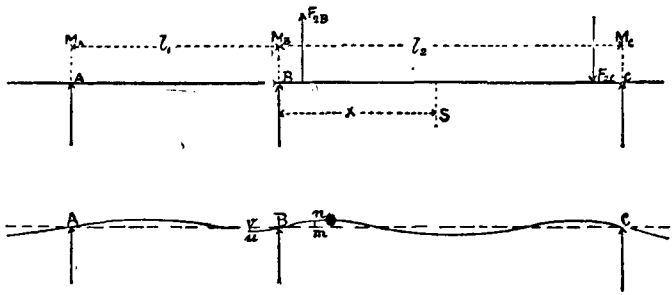

Fig. 39.

intermediate spans, and w_1 w_2 the loads upon them per foot run uniformly distributed.

The distinctions between forces of opposite sense must be observed carefully, adopting the customary convention, + sign will indicate a force acting upwards, and — sign a force acting downwards. Thus P_A P_B P_C the reactions at the supports will all be of negative sign.

Moments will take sign according as they tend to produce convexity in the beam, upwards or downwards. Thus, if the mean fibre of the beam take the form shown below in the figure ; M_A M_B M_C will be positive, and bending moments near the middle of each span negative.

However, since in any cross section of a beam, the bending moment and shearing on one side, are opposite in sense to those on the other, it

is necessary to premise *that it is always the right hand side of a section that is considered*, unless the reverse is expressly stated.

The formula is proved by determining the inclination of the mean fibre at the point B, first by the point of view from l_2, then by the point of view from l_1 and equating the two, *i.e.*, the angle m B n = the angle u B v.

These angles are formed by the same method, which was adopted in Chapters XVIII, XIX, XX, XXI, etc., for determining deflection, namely, by substituting for M in the formula

$$d\,\frac{dy}{dx} = \frac{M}{E\,I}\,dx\,;$$

the general expression for bending moment on the span. Hitherto it has been assumed, that the reader is able to obtain this expression for himself, but in this case, though the principle is the same, the circumstances are peculiar, and the process will therefore be detailed.

First imagine the span of bean, l_2 to be cut through at both ends as close as possible to the supports B and C respectively ; consider then its equilibrium. At the extremity B it is acted upon by the moment M_B (with a + sign), at the extremity C it is acted upon by the moment — Mc of minus sign because the left hand side of the section is here considered ; in addition, there are the shearing forces + F_{2B} and — F_{2B} acting at either end respectively, as well as the uniform load, w_2 lb. per foot run.

Hence taking moments about B_1 we have :

$$M_B - M_C - F_{2C}\ l_2 + \frac{w_2}{2}\,l_2^2,$$

and since the span of beam, considered is in equilibrium, this expression is equal to zero

$$M_B - M_C + \frac{w_2}{2}\,l_2^2 - F_{2C}\,l_2 = 0.$$

Whence $$F_{2C} = \frac{1}{l_2}\,(M_B - M_C) + \frac{w_2}{2}\,l_2.$$

Now take another section in the span l_2 at any point S distance x from B, and consider the equilibrium of the piece of beam S B in exactly the same way.

It will be found,

$$M - M_C + \frac{w_2}{2}\,(l_2 - x)^2 - F_{2C}\,(l_2 - x) = 0,$$

here M the bending moment on the section at S, takes the place of M_B and $(l_2 - x)$ the length of beam considered takes the place of l_2 in the former case.

But $$F_{2C} = \frac{1}{l_2}\,(M_B - M_{2C}) + \frac{w_2}{2}\,l_2.$$

Therefore

$$M = M_C - \frac{w_2}{2}(l_2 - x)^2 + \left(\frac{M_B}{l_2} - \frac{M_C}{l_2} + \frac{w_2}{2} l_2\right)(l_2 - x)$$

which is the general expression for bending moment at any point in the span l_2 distance x from B.

A more convenient form is obtained by expanding.

Thus $M = M_C - \frac{w_2}{2} l_2^2 + w_2 l_2 x - \frac{w_2}{2} x^2 + M_B - M_C + \frac{w_2}{2} l^2$

$$- \frac{M_B}{l_2} x + \frac{M_C}{l_2} x - \frac{w_2}{2} l_2 x$$

$$= M_B + (M_C - M_B)\frac{x}{l_2} + \frac{w_2}{2} l_2 x - \frac{w_2}{2} x^2.$$

Introducing this value of M into the formula

$$d\frac{dy}{dx} = \frac{M}{E I} dx;$$

the differential equation for the deformed mean fibre proved in Chapter XVII.

We have

$$d\frac{dy}{dx} = \left(M_B + (M_C - M_B)\frac{x}{l_2} + \frac{w_2}{2} l_2 x - \frac{w_2}{2} x^2\right)\frac{dx}{E I}.$$

In the previous chapters on deflections, the method of which we are now following, our origin (*i.e.*, the point from which x is measured), was always taken at a place, where the mean fibre would remain level, so that when $x = 0$, then $\frac{dy}{dx} = 0$. Now, however, the origin being taken at the extremity of the beam, when x is zero, $\frac{dy}{dx}$ is the inclination (or rather the *tan* of the inclination) of the mean fibre at B; so that when the right hand side of the equation is integrated from 0 to x, $\frac{dy}{dx}$ must be integrated from *tan* θ to $\frac{dy}{dx}$.

Hence

$$E I \int_{\tan\theta}^{\frac{dy}{dx}} d\frac{dy}{dx} = \int_0^x \left(M_B dx + \frac{M_C - M_B}{l_2} x\, dx + \frac{w_2}{2} l_2 x\, dx - \frac{w_2}{2} x^2 dx\right),$$

or

$$E I \left(\frac{dy}{dx} - \tan\theta\right) = M_B x + \frac{M_C - M_B}{l_2} \cdot \frac{x^2}{2} + \frac{w_2}{2} l_2 \frac{x^2}{2} - \frac{w_2}{2} \cdot \frac{x^3}{3}$$

or $E I \frac{dy}{dx} = E I \tan\theta + M_B x + \frac{M_C - M_B}{l_2} \cdot \frac{x^2}{2} + \frac{w_2}{4} l_2 x^2 - \frac{w_2}{6} x^3.$

$$\text{Again } E\,I \int_0^y dy = \int_0^x \left(E\,I \tan\theta\, dx + M_B x\, dx + \frac{M_C - M_B}{l_2} \cdot \frac{x^2}{2} dx + \frac{w_2}{4} l_2 x^2\, dx - \frac{w_2}{6} x^3\, dx\right),$$

or

$$y\,E\,I = E\,I \tan\theta\, x + M_B \frac{x^2}{2} + \frac{M_C - M_B}{l_2} \cdot \frac{x^3}{6} + \frac{w_2}{4} l_2 \frac{x^3}{3} - \frac{w_2}{6} \cdot \frac{x^4}{4}.$$

When x is made equal to l_2, y becomes zero, being then the deflection at the point of support C.

Hence

$$0 = E\,I \tan\theta\, l_2 + M_B \frac{l_2^2}{2} + \frac{M_C - M_B}{l_2} \cdot \frac{l_2^3}{6} + \frac{w_2}{12} l_2^4 - \frac{w_2}{24} l_2^4,$$

or

$$-E\,I \tan\theta = M_B \frac{l_2}{3} + M_C \frac{l_2}{6} + \frac{w_2}{24} l_2^3.$$

Applying exactly the same process of reasoning to span l_1 we obtain

$$-E\,I \tan\theta = -M_B \frac{l_1}{3} - M_A \frac{l_1}{6} - \frac{w_1}{24} l_1^4$$

Therefore $M_B \frac{l_2}{3} + M_C \frac{l_1}{6} + \frac{w_2}{24} l_2^3 = -M_B \frac{l_1}{3} - M_A \frac{l_1}{6} - \frac{w_1}{24} l_1^3$

or $$M_A \frac{l_1}{6} + M_B \left(\frac{l_2 - l_1}{3}\right) + M_C \frac{l_1}{6} = -\frac{w_1}{24} l_1^3 - \frac{w_2}{24} l_2^3$$

Whence $$4\,M_A l_1 + 8\,M_B (l_1 + l_2) + 4\,M_C l_2 = -w_1 l_1^3 - w_2 l_1^3.$$

The minus sign appears on the left hand side, because w the load being a force acting downwards, is negative as explained before.

A girder may be continuous over any number of supports ; but the bending moment over the first and last ones can be obtained at once in the ordinary way, then applying the above formula to each three consecutive supports in order, thus A B C, B C D, C D E, etc., a number of simultaneous equations are obtained, sufficient to give the bending moment at every pier.

COROLLARY :—(1.)—M_B and M_C having been determined as above, we may find the bending moment at any point in the span by substituting their numerical values in the equation,

$$M = M_B + (M_C - M_B) \frac{x}{l_1} + \frac{w_2}{2} l_2 x - \frac{w_2}{2} x^2$$

There are generally two points in each span, called the points of inflection, where the bending-moment is nothing; these may be determined by putting $M = 0$ in the above general equation.

Then $$x^2 \frac{w_2}{2} - x \left\{ (M_C - M_B) \frac{1}{l_2} + \frac{w_2}{2} l_2 \right\} - M_B = 0$$

And solving

$$x = \frac{1}{w_2}\left\{(M_C - M_B)\frac{1}{l_2} + \frac{w_2}{2}\,l_2 \pm \sqrt{\left\{(M_C - M_B)\frac{1}{l_2} + \frac{w_2}{2}\,l_2\right\}^2 + l\,w_2\,M_B}\right\} \quad \ldots \quad (1.)$$

EXAMPLE :—In the case of a very long continuous beam supported in very many places at equal distances apart, and subject to the same uniform load throughout; the bending moment over three adjacent supports near the middle of the beam would be nearly the same.

Let M_P denote their common values.

Then the equation $4\,M_A\,l_1 + 8\,M_B\,(l_1 + l_2) + 4\,M_C\,l_2 = -w_1\,l_1^3 - w_2\,l_2^3$

becomes $\qquad M_P\,l + 4\,M_P\,l + M_P\,l = -\frac{1}{2}\,w\,l^3.$

and $$M_P = -\frac{w}{12}\,l^2$$

Also the expression (1) above becomes

$$\frac{1}{w}\left\{\frac{w}{2}\,l \pm \sqrt{\left(\frac{w}{2}\,l\right)^2 + l\,w\,M_P}\right\}$$

$$= \frac{1}{w}\left\{\frac{w}{2}\,l \pm \sqrt{\frac{w^2}{4}\,l^2 + \frac{w^2}{6}\,l^2}\right\}$$

$$= l\left(\tfrac{1}{2} \pm \sqrt{\tfrac{1}{12}}\right)$$

Therefore $\qquad x = \cdot 7887l$ and $\cdot 2113l$

which are the distances of the points of inflection from the end of the span.

NOTE :—With aid of these facts, we are able to estimate the strength of a beam encastré (completely fixed) at both ends ; for the condition of such a beam, with respect to its loading does not differ from that of one of the middle spans in the long girder described in the example ; also the length of beam between the two points of inflection (amounting to $\cdot 5774l$.), will be in the same condition as an ordinary beam of the same length simply supported at its extremities.

Hence, a beam encastré at both ends is nearly as strong as a similar beam of only half the length, if the latter is merely supported at each end.

COROLLARY :—(2.) P_A P_B, the reactions at the supports will evidently be equal to the difference of the shearing forces on sections immediately on either side of each ; since it can only be the upward force supplied by a support, which produces the change of shearing forces, in passing over from one side to the other.

Thus $$P_B = F_{2B} - F_{1B},$$

it should be observed that the shearing forces being opposite in sign are numerically added.

F_{1B} and F_{2B} can be found in the same manner that F_{2C} was obtained. Or simply altering the suffixes, B to A, C to B, and 2 to 1, we have from the expression for F_{2C},

$$F_{1B} = \frac{1}{l_1}(M_A - M_B) + \frac{w_1}{2} l_1$$

and by symmetry, $$-F_{2B} = \frac{1}{l_2}(M_C - M_B) + \frac{w_2}{2} l_2$$

Adding these we obtain

$$P_B = \frac{1}{l_1}(M_A - M_B) + \frac{1}{l_2}(M_C - M_B) + \frac{w_1}{2} l_1 + \frac{w_2}{2} l_2.$$

EXAMPLE :—A beam of length $2\,l_2$ uniformly loaded, is supported in the centre and at each end ; it is required to find the bending moment over the middle support, also the proportion of weight sustained by each support.

In the formula $4\,M_A + 8\,M_B\,(l_1 + l_2) + 4\,M_C = -w_1 l_1^3 - w_2 l_2^3$ $l_1 = l_2$ and M_A M_C are both nothing.

So the formula becomes, $16\,M_B\,l = -2\,w\,l^3$

Whence $$M_B = \frac{w\,l^2}{8}$$

this is the bending moment over the central support.

Again the formula

$$P_B = \frac{1}{l_1}(M_A - M_B) + \frac{1}{l_2}(M_C - M_B) + \frac{w_1}{2}\,l_1 + \frac{w_2}{2} l_2$$

becomes $$P_B = -l\,\frac{M_B}{l} + w\,l$$

$$= \tfrac{3}{4}\,w\,l_1 \text{ because } M_B = \frac{w\,l^2}{8}.$$

Now the total load is $2\,w\,l$ and since $\frac{3}{4}\,w\,l$ is carried by the central support, there remains $1\frac{1}{4}\,w\,l$ to be divided equally between the other two, which therefore carry $\frac{5}{8}\,w\,l$ each. This distribution of load is often of importance in the design of floors and roof.

NOTE.—Throughout this chapter, I the moment of inertia of the beam has been assumed constant.

XXV.—Expansion Rollers.

Formula :— $$W^2 = \frac{64}{9} p_o^3 \frac{l^2 R^2}{E},$$

where l is the length of the roller and R its radius. Also E is the modulus of elasticity, and p_o the greatest intensity of stress which obtains.

Fig. 40 represents the roller resting on a plate at 0 ; x is the distance of any point in the curve X 0 X¹, from the *diameter* of the roller through 0, and y is its distance from the *tangent* at 0.

In the thirty-fifth proposition of the third book of Euclid, the words of the writer are thus rendered :—

"If through any point within a circle, two straight lines be drawn, and terminated at the circumference of that circle ; then the rectangle contained by the two segments of one of them, shall be equal to the rectangle contained by the segments of the other."

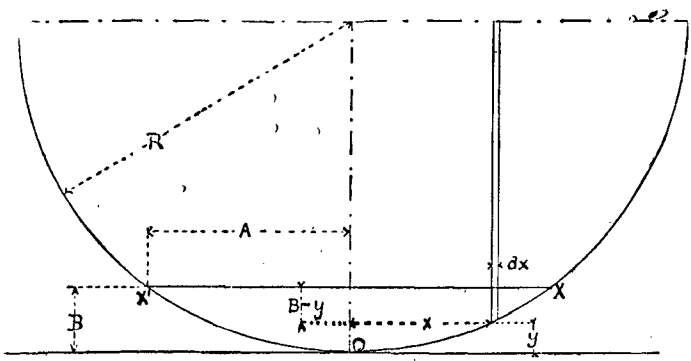

Fig. 40.

Therefore $$x^2 = (2\,R - y)\,y = 2\,R\,y - y^2,$$

but y^2 being of necessity an extremely small quantity, may be neglected without injury.

Hence $$x^2 = 2\,R\,y \text{ ; or } y = \frac{x^2}{l\,R}.$$

So far geometrically.

Let straight line X 0 X¹ represent the flattening which takes place under loading.

It will be seen, that the weight upon the roller may be regarded as sustained by a number of very thin laminæ, or plates, (of length l perpendicular to the paper and of thickness dx. As the breadth of flattening

that can be permitted in practice is very little indeed, these plates or laminæ may all be regarded as originally of the same height 2 R. Also since the flattening which takes place at the bottom of the roller must occur at the top as well, the *horizontal* diameter will remain unaffected, and it will be sufficient to consider only the lower half of these laminæ.

After loading, the strain upon each of them is

$$\frac{B - y}{R},$$

and therefore the stress is $E \frac{B - y}{R}$ (see Chapter I.), this stress reaches its maximum in the middle where $y = o$.

Hence the greatest stress is $p_o = E \frac{B}{R}$.

To find the actual pressure exerted by one of the supposed vertical laminæ, we must multiply *its area* by the stress upon it;

whence we get $$E \frac{B - y}{R} l\, dx.$$

The total amount of all these small pressures may be summed up by integration, by giving x all the values from 0 to A twice over, and must equal the whole load upon the roller.

Thus :—

$$W = 2 \int_o^A E \frac{B - y}{R} l\, dx$$

$$= 2 l \frac{E}{R} \int_o^A \left(B\, dx - \frac{x^2}{2R}\, dx \right). \quad \text{For } y = \frac{x^2}{2R}$$

$$= 2 l \frac{E}{R} \left(B A - \frac{A^3}{6R} \right).$$

Now putting $x = A$; and $y = B$ in the equation $x^2 = 2 R y$, we have $A^2 = 2 R B$, or $A = \sqrt{2 R B}$.

Therefore $$W = 2 l \frac{E}{R} \left(B \sqrt{2 R B} - \sqrt{\frac{(2 R B)^3}{6 R}} \right),$$

or $$W^2 = 4 l^2 \frac{E^2}{R^2} \left\{ B^2 (2 R B) - \frac{2 B (2 R B)^2}{6 R} + \frac{(2 R B)^3}{36 R^2} \right\}$$

$$= 4 l^2 \frac{E^2}{R^2} \left(2 - \frac{4}{3} + \frac{2}{9} \right) R B^3.$$

Now $p_o = E \frac{B}{R}$; where p_o is the greatest stress that obtains in the roller. Therefore $B = p_o \cdot \frac{R}{E}$.

Hence $$W^2 = 4\, l^2 \frac{E^2}{R} \left(\frac{8}{9}\right) p_o^3 \frac{R^3}{E^3}$$

$$= \frac{32}{9}\, p_o^3\, \frac{l^2\, R^2}{E}.$$

The flattening of the roller has been represented by the *straight* line XX^1, this could only be correct if the bed plate was perfectly rigid and unyielding, or if instead of the bed plate there was substituted another roller of the same diameter.

Under ordinary circumstances the line XX^1 would be curved downwards; the intensity of pressure in the middle is thus reduced to an indeterminate degree, and the formula modified to allow for this condition, becomes

$$W^2 = \frac{64}{9}\, p_o^3\, \frac{l^2\, R^2}{E}.$$

COROLLARY :— $$R = \frac{3}{8\, l} \sqrt{\frac{W\, E}{p_o^3}}.$$

WERTHEIMER, LEA & Co., Printers, Circus Place, London Wall, E.C.

NOTES.

NOTES.

NOTES.

NOTES

NOTES.

NOTES.

NOTES.

NOTES.

00063762

Digitized with financial assistance from
Observer Research Foundation
on 22 February, 2019